Umans Complexity Theory Lectures

Lecture 1e: Deterministic Hierarchy Theorems

Complexity Classes

- Back to complexity classes:
  - **TIME(T(n))** = languages decidable by a multi-tape TM in at most T(n) steps, where n is the input length, and T :**N** → **N**
  - **SPACE(S(n))** = languages decidable by a multi-tape TM that touches at most S(n) squares of its work tapes, where n is the input length, and S :**N** → **N**

Note: $\mathbf{P} = \cup_{k \geq 1} \mathbf{TIME}(n^k)$

Hierarchy Theorems

- Does **genuinely** more time permit us to decide new languages?
- how can we construct a language L that is *not* in **TIME(T(n))…**
  - idea: same as "HALT undecidable" diagonalization and simulation

Hierarchy Theorems

- Does **genuinely** more space or time permit us to decide new languages?

Theorem (Space Hierarchy Theorem): For every *proper complexity function* S(n) ≥ log n: **SPACE(o(S(n))) ⊊ SPACE(S(n)).**

- Proof: Diagonalization

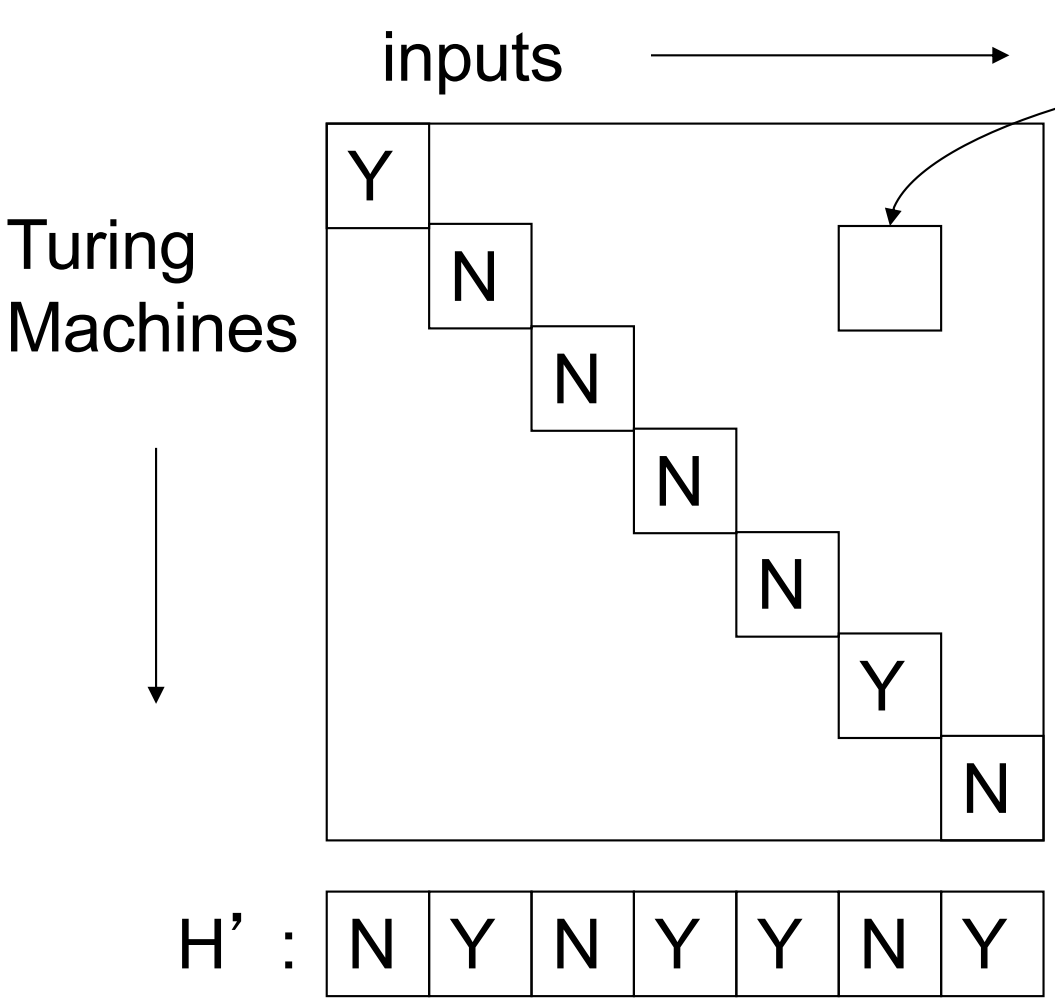
Theorem (Time Hierarchy Theorem): For every *proper complexity function* T(n) ≥ n:

TIME(T(n)) ⊊ TIME(T(n)²) (for 1 tape TMs)

TIME(T(n)) ⊊ TIME(T(n) log T(n)) (for multitape TMs)

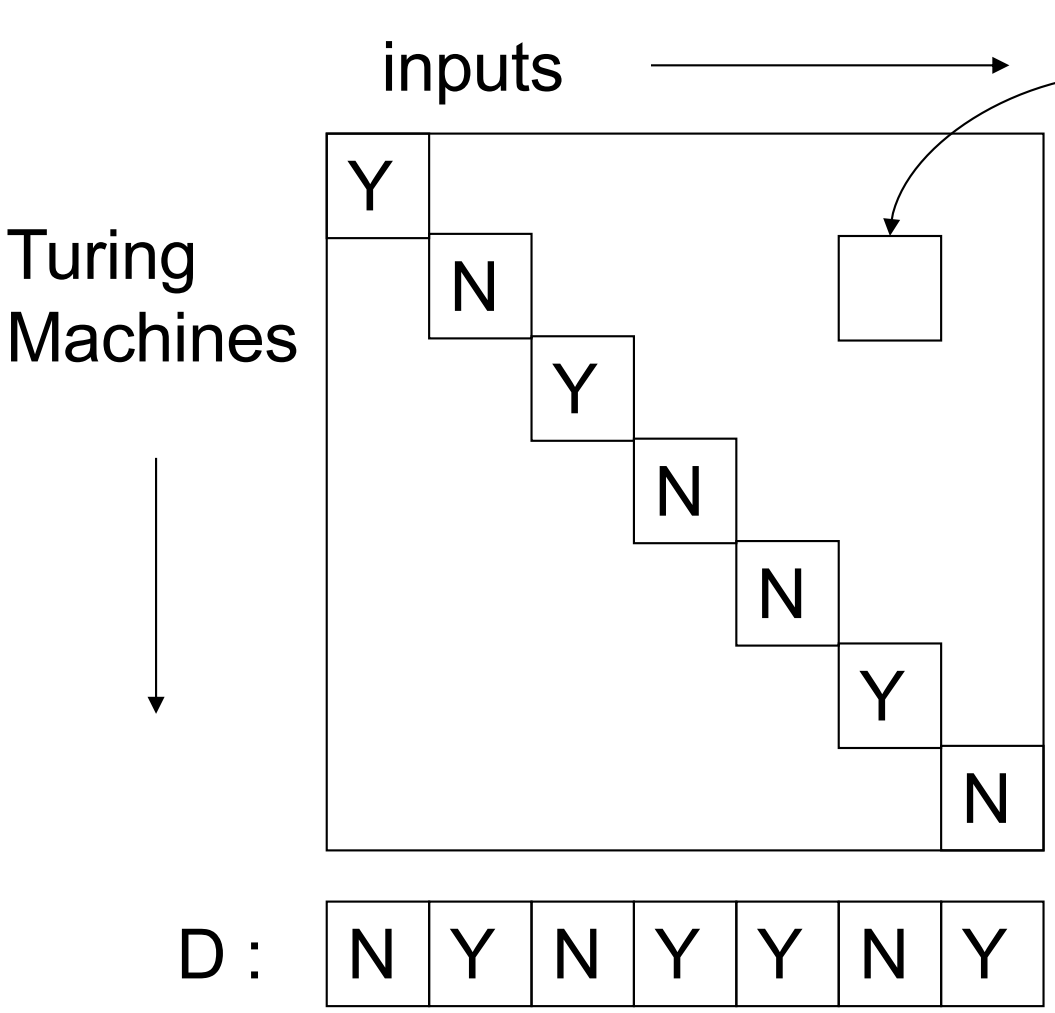
- Proofs: Diagonalization

Recall proof for Halting Problem



The existence of H which tells us yes/no for each box allows us to construct a modified TM H' that cannot be in the table.

Deterministic Time Hierarchy Theorem



- TM simulation SIM tells us yes/no for each box **in time T'(n)**
- rows include all of **TIME(T(n))**
- construct TM D running in time T'(2n) that is not in table

DeterministicTime Hierarchy Theorem

Theorem (Deterministic Time Hierarchy Theorem): For every *proper complexity function* T(n) ≥ n:

TIME(T(n)) ⊊ TIME(T(n)^2) for 1 tape TMs.

We will just prove:

TIME(T(n)) ⊊ TIME(T(2n)^3) for 1 tape TMs.

Proof of Deterministic Time Hierarchy Theorem

Proof: Assume (for sake of contradiction):

TIME(T(n)) = TIME(T(2n)^3) for 1 tape TMs.

– SIM is universal TM deciding language

{ <M, x> : M accepts x in ≤ T(|x|) steps }

– Claim (proved later): SIM runs in time T'(n) = T(n)^3.

– Define new TM D: on input <M>

- if SIM accepts <M, M>, reject
- if SIM rejects <M, M>, accept

– So M(<M>) = SIM(<M, M>) ≠ D(<M>)

– D runs in time T'(2n).

Proof of Deterministic Time Hierarchy Theorem

- Proof (continued):
  - suppose M in **TIME(T(n))** decides L(D)
    - M(<M>) = SIM(<M, M>) ≠ D(<M>)
    - but M(<M>) = D(<M>)
  - Contradiction !

Proof of Deterministic Time Hierarchy Theorem

- Claim: there is a TM SIM that decides

  {<M, x> : M accepts x in ≤ T(|x|) steps}

  and runs in time $T'(n) = T(n)^3$.
- Proof sketch: SIM is universal TM with 4 work tapes:
  - contents and "virtual head" positions for M's tapes
  - M's transition function and state
  - Constructable time bound T(|x|) "+"s used as a clock
  - scratch space

Proof of Deterministic Time Hierarchy Theorem

  - contents and "virtual head" positions for M's tapes
  - M's transition function and state
  - T(|x|) "+"s used as a clock
  - scratch space
- initialize tapes
- simulate step of M, advance head on tape 3; repeat.
- can check running time is as claimed.

- Important detail: need to initialize tape 3 in time O(T(n))

Relationships between complexity classes

- Time Hierarchy Theorem implies

  $P \subsetneq EXP$

  – $P \subsetneq TIME(2^n)$ **(** $TIME(2^{(2n)3}) \subseteq EXP$

- Space Hierarchy Theorem implies

  $L \subsetneq PSPACE$

  – $L = SPACE(\log n) \subsetneq SPACE(\log^2 n) \subsetneq PSPACE$