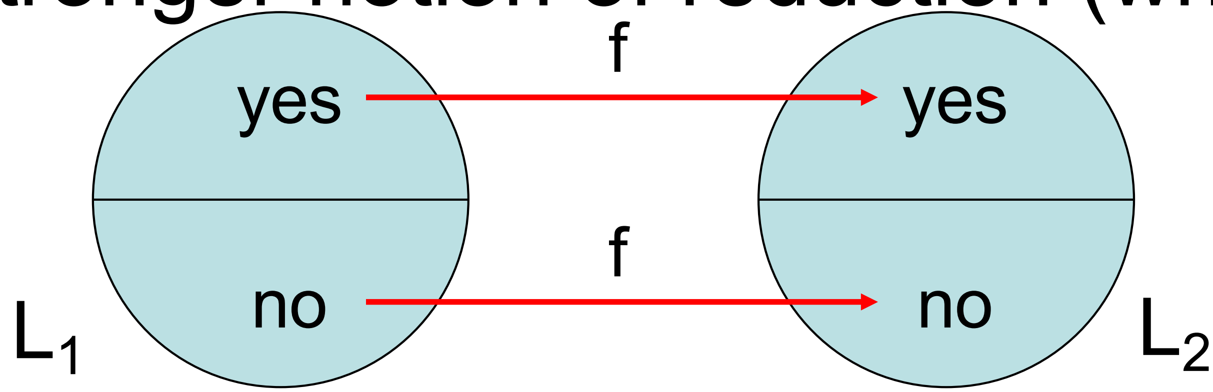
# Umans Complexity Theory Lectures

## Lecture 2b: A P-Complete Problem

# A **P**-complete problem

- P = class of languages accepted in polytime
- L = class of languages accepted by logspace

- We don't know how to prove **L** ≠ **P**
- But, can identify problems in **P** least likely to be in **L** using **P**- completeness.
- need stronger notion of reduction (why?)



# A **P**-complete problem

- **logspace reduction**: function f computable by TM that uses O(log n) space
  - denoted "$L_1 \leq_{\mathbf{L}} L_2$"
- **polytime reduction**: function f computable by TM that uses $n^{O(1)}$ time
  - denoted "$L_1 \leq_{\mathbf{P}} L_2$"

- If $L_2$ is **P**-complete, then $L_2$ in **L** implies **L** = **P**

# A **P**-complete problem

- Circuit Value (CVAL): given a variable-free Boolean circuit (gates $\wedge$, $\vee$, $\neg$, 0, 1), does it output 1?

**Theorem**: CVAL is **P**-complete.

- Proof:
  - already argued in **P**
  - L arbitrary language in **P,** TM M decides L in $n^c$ steps

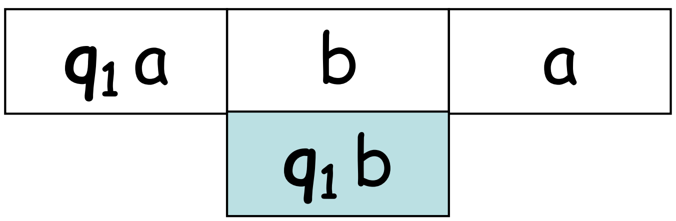
# A **P**-complete problem

- Tableau (configurations written in an array) for machine M on input w:
  - height = time taken = $|w|^c$
  - width = space used ≤ $|w|^c$

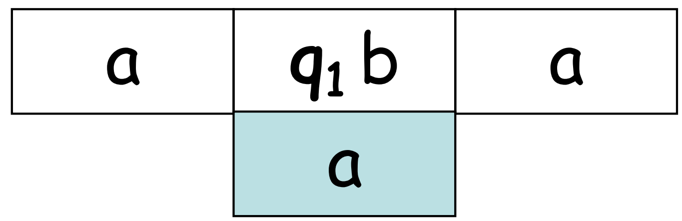
| | | | | | | |
|---|---|---|---|---|---|---|
| $q_s w_1$ | $w_2$ | … | $w_n$ | … | _ | Initial Config |
| $w_1$ | $q_1 w_2$ | … | $w_n$ | … | _ | Move Right |
| $q_1 w_1$ | $a$ | … | $w_n$ | … | _ | Move Left |
| | ⋮ | | | | ⋮ | |
| $q_a$ _ | _ | … | _ | … | _ | Halt |

# A **P**-complete problem

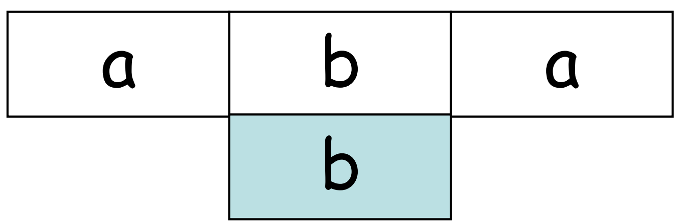
- Important observation: contents of cell in tableau determined by 3 others above it:




Head moves right



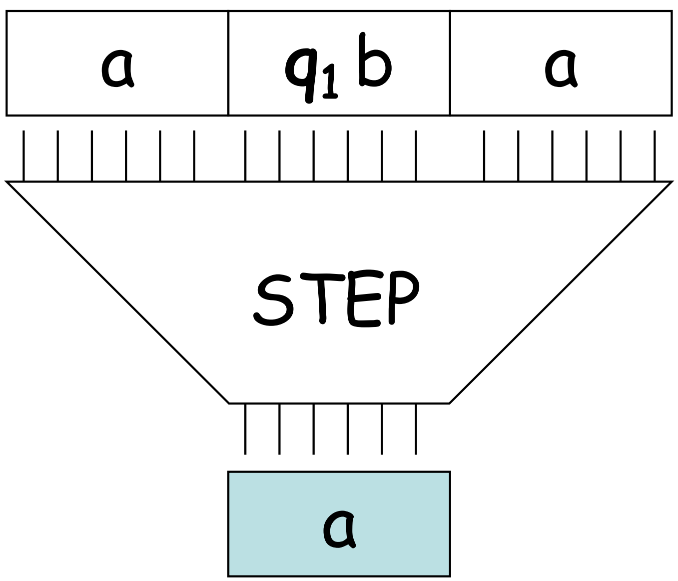

Head writes an a and moves




No Head at tape cell,
so cell does not change

# A **P**-complete problem

- Can build Boolean circuit STEP
  - input (binary encoding of) 3 cells
  - output (binary encoding of) 1 cell




- each output bit is some function of inputs
- can build subcircuit for each step
- size of entire circuit is linear in size of tableau

# A **P**-complete problem

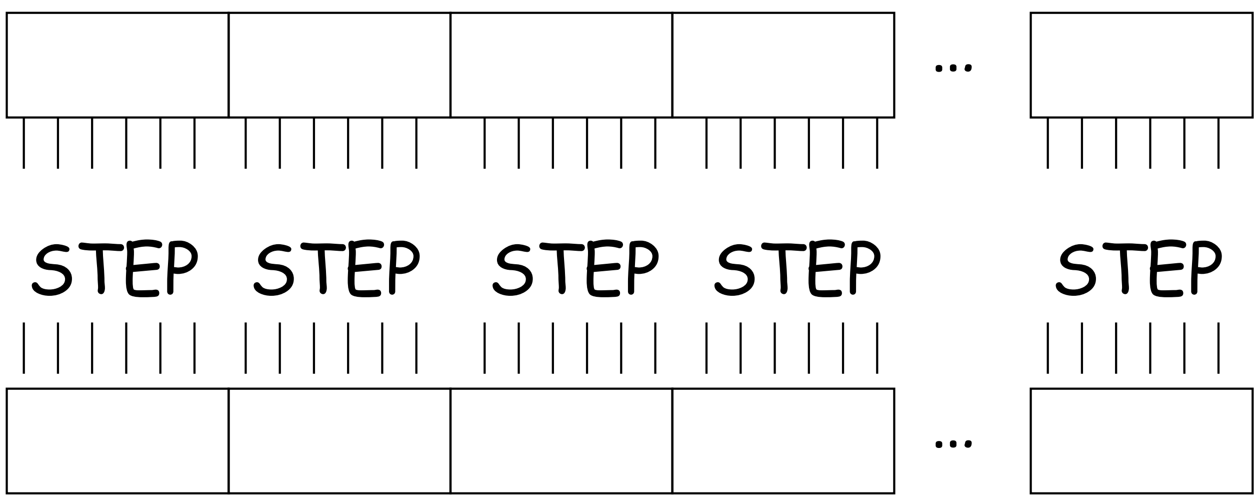
Tableau for M on input

$w = w_1 w_2 \dots w_n$

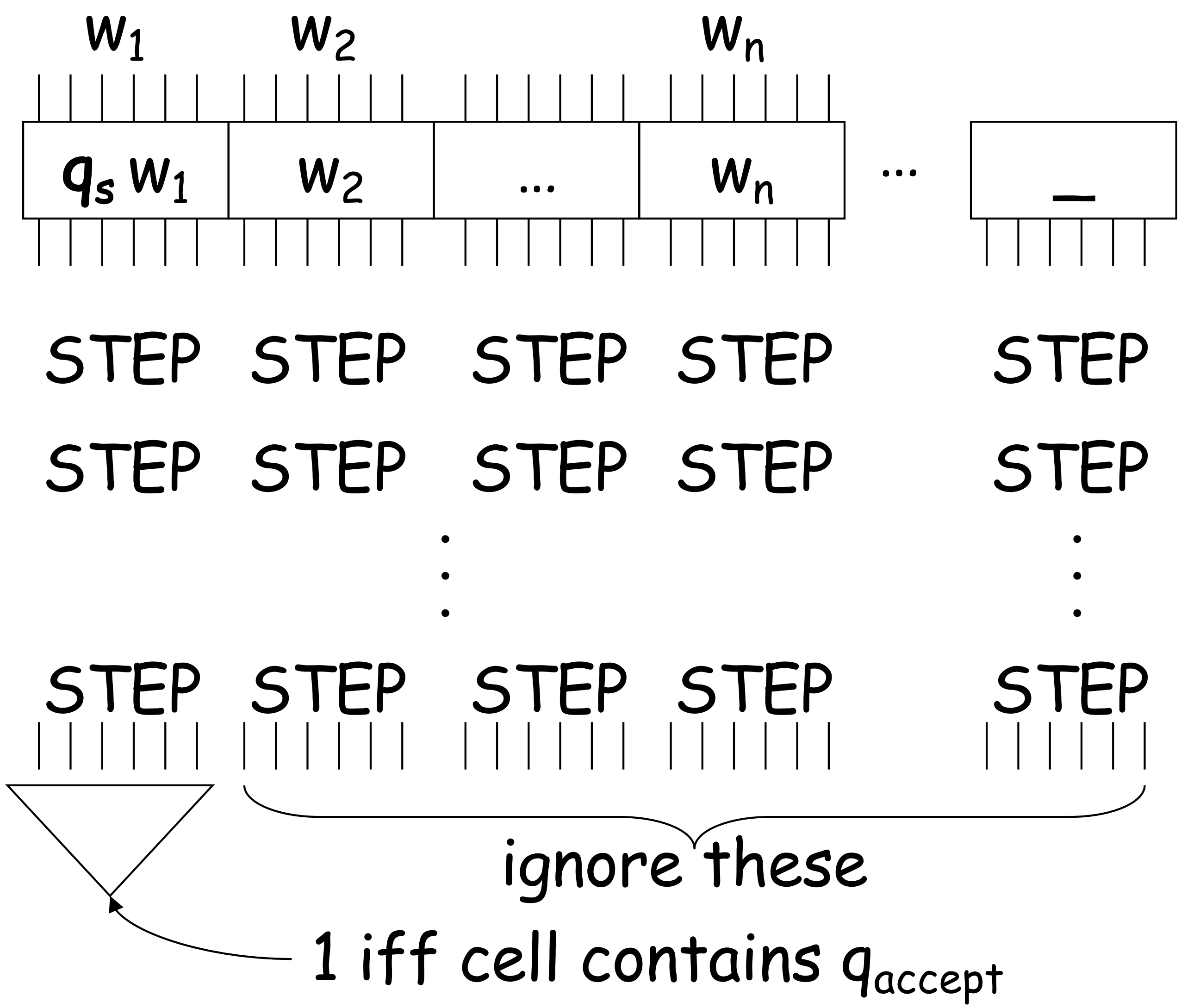
| | | | | | |
|---|---|---|---|---|---|
| $q_s w_1$ | $w_2$ | … | $w_n$ | … | _ |
| $w_1$ | $q_1 w_2$ | … | $w_n$ | … | _ |

$\vdots \qquad \vdots$

- $|w|^c$ copies of STEP compute row i from i-1



# A **P**-complete problem




This circuit $C_{M,\,w}$ has input $w_1w_2\ldots w_n$

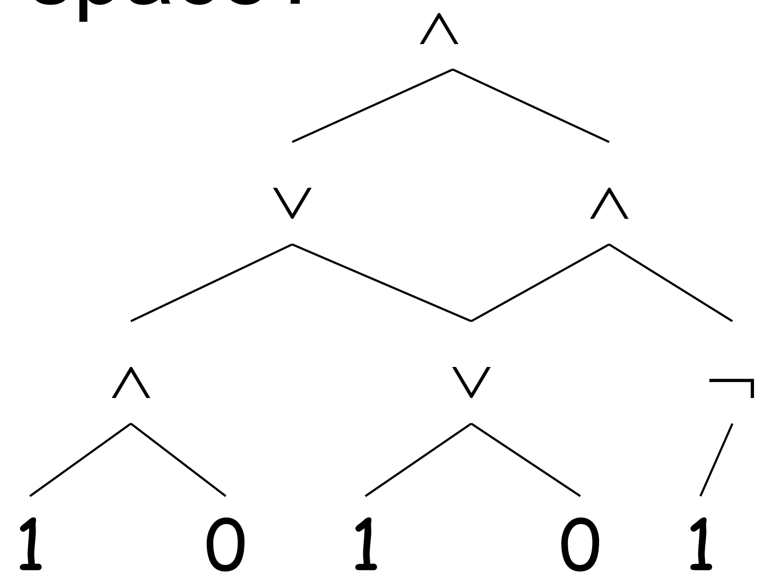
$C(w) = 1$ iff M accepts input w.

logspace reduction

Size = $O(|w|^{2c})$

# Answer to question

- Can we evaluate every n node Boolean **circuit** using O(log n) space?




- NO! (probably)

- CVAL in **L** if and only if **L** = **P**