Umans
Complexity Theory Lectures

Lecture 4b:

Nondeterministic Space:

I-S Theorem: NL = coNL

Nondeterministic Space

- nondeterminism applied to space
- reachability
- two surprises:
  - Savitch's Theorem
  - Immerman/Szelepcsényi Theorem

Nondeterministic space

- **NSPACE(f(n))** = languages decidable by a multi-tape NTM that touches at most f(n) squares of its work tapes *along any computation path*, where n is the input length, and f :$\mathbf{N} \rightarrow \mathbf{N}$

Nondeterministic space

- Robust nondeterministic space classes:

NL = NSPACE(log n)

NPSPACE = $\cup_k$ **NSPACE(n^k)**

Summary

- nondeterministic space classes

  NL and **NPSPACE**

- ST-CONN **NL**-complete

Summary

- Savitch: **NPSPACE = PSPACE**
  - Proof: ST-CONN ∈ **SPACE($\log^2$ n)**
  - open question:

    NL = L?

- Immerman/Szelepcsényi : **NL = coNL**
  - Proof: ST-NON-CONN ∈ **NL**

Second startling theorem

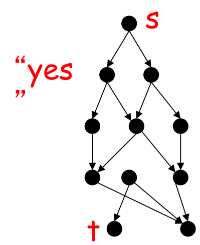
- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal
- for space: Immerman/Szelepscényi '87/'88:

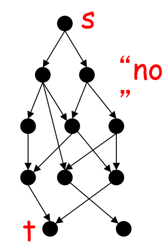
  NL = coNL

I-S Theorem

Theorem: ST-NON-CONN ∈ **NL**

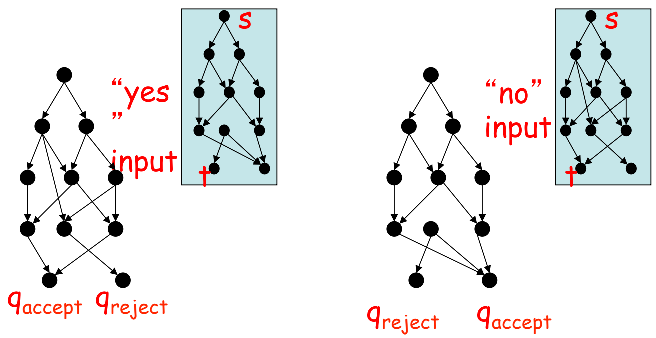
- Proof: slightly tricky setup:
  - input: G = (V, E), two nodes s, t





I-S Theorem

- want nondeterministic procedure using only O(log n) space with behavior:



I-S Theorem

- observation: given **count** of # nodes reachable from s, can solve problem
  - Initialize a counter to 0
  - for each v $\in$ V, *guess* if v is reachable from s and if yes, *guess* path from s to v
    - if guess doesn't't lead to v, reject.
    - if v = t, reject.
    - else increment counter by 1
  - if counter = **count** accept

I-S Theorem

- every computation path has sequence of guesses…
- only way computation path can lead to accept:
  - correctly guessed reachable/unreachable for each node v
  - correctly guessed path from s to v for each reachable node v
  - saw *all* reachable nodes, and
  - t not among reachable nodes

I-S Theorem

- R(i) = # nodes reachable from s in at most i steps
- R(0) = 1: node s

- we will compute R(i+1) from R(i) using O(log n) space and nondeterminism

- computation paths with "bad guesses" all lead to reject

I-S Theorem

– Outline: in n phases, compute

R(1), R(2), R(3), … R(n)

– only O(log n) bits of storage between phases

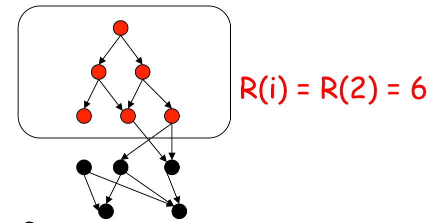
– in end, lots of computation paths that lead to reject

– only computation paths that survive have computed correct value of R(n)

– apply observation.

I-S Theorem

– computing R(i+1) from R(i):



– Initialize R(i+1) = 0

– For each v ∈ V, *guess* if v reachable from s in at most i+1 steps

I-S Theorem

– if "yes", *guess* path from s to v of at most i+1 steps. Increment R(i+1)

– if "no", visit R(i) nodes reachable in at most i steps, check that none is v or adjacent to v

 –for u ∈ V *guess* if reachable in ≤ i steps; *guess* path to u; increment counter by 1

- KEY: if counter ≠ R(i), reject
- at this point: **can be sure v not reachable**

I-S Theorem

- correctness of procedure:
- two types of errors we can make
- (1) might guess v is reachable in at most i +1 steps when it is not

 – won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

I-S Theorem

- (2) might guess v is **not** reachable in at most i+1 steps when it is
  - then must **not** see v or neighbor of v while visiting nodes reachable in i steps.
  - but forced to visit R(i) distinct nodes
  - therefore must try to visit node v that is **not** reachable in ≤ i steps
  - won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error