Quantum Computing
Lecture 14a

(notes on QEC)

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Classical Error Correcting Codes

- Suppose errors in our physical system for storing 0 and 1 cause each physical bit to be toggled independently with probability $p$
- We can reduce the probability of error to be in $O(p^2)$ by using a “repetition code”
- e.g. : encode a logical 0 with the state 000 and a logical 1 with the state 111
Reversible networks for encoding and decoding
Classical Error Correcting Codes

- After the errors occur, decode the logical bits by taking the majority answer of the three bits and correct the encoded bits

- So

  - 000 → 000
  - 001 → 000
  - 010 → 000
  - 100 → 000
  - 111 → 111
  - 011 → 111
  - 101 → 111
  - 110 → 111
Classical Error Correcting Codes

- As long as less than 2 errors occurred, we will keep the correct value of the logical bit.
- The probability of 2 or more errors is

\[ 3p^2(1 - p) + p^3 = 3p^2 - 2p^3 \in O(p^2) \]

(which is less than p if \( p < \frac{1}{2} \))
Reversible network for error correction

- Assume that \( e_3 + e_2 + e_1 \leq 1 \) where \( e_i \in \{0,1\} \)

- If \( s_1s_2 = 00 \) then no error occurred
- Otherwise, the error occurred in bit \( j \) where \( j = 2s_1 + s_2 \)
Equivalently
Stabilizer measurement??

This is implementing a $Z_1$ measurement (interpreting 0 as +1, and 1 as -1)
Stabilizer measurement??

This is implementing a $Z_1Z_2$ measurement
Stabilizer measurement??

This is implementing a $X_1X_2$ measurement
Notation clarification

- For an n-qubit system $Z_j$ denotes

\[
\underbrace{I \otimes I \otimes \cdots \otimes I}_{j-1} \otimes Z \otimes \underbrace{I \otimes \cdots \otimes I}_{n-j}
\]

- E.g. $n=3$, then

\[
Z_1 Z_2 = (Z \otimes I \otimes I)(I \otimes Z \otimes I) = (Z \otimes Z \otimes I)
\]
Perform operations on logical bits

- e.g. NOT gate

\[b \quad \overline{X} \quad \overline{b}
\]

\[b \quad \overline{X} \quad \overline{b}
\]

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\]

\[b \quad \overline{X} \quad \overline{b}
\]
Perform operations on logical bits

- e.g. c-NOT gate
Quantum Error Correcting Codes

- e.g. : encode a logical $|0\rangle$ with the state $|000\rangle$
  and a logical $|1\rangle$ with the state $|111\rangle$
Quantum network for encoding

\[ (\alpha |0\rangle + \beta |1\rangle) |0\rangle |0\rangle \rightarrow \alpha |0\rangle |0\rangle |0\rangle + \beta |1\rangle |1\rangle |1\rangle \]
Quantum network for correcting errors

- Assume that $e_3 + e_2 + e_1 \leq 1$ where $e_i \in \{0,1\}$

\[
\begin{align*}
\alpha | e_3 \rangle | e_2 \rangle | e_1 \rangle + \beta | 1 \oplus e_3 \rangle | 1 \oplus e_2 \rangle | 1 \oplus e_1 \rangle & \rightarrow \\
\alpha | 0 \rangle | 0 \rangle | 0 \rangle + \beta | 1 \rangle | 1 \rangle | 1 \rangle
\end{align*}
\]
Equivalently
Perform operations on logical bits

- e.g. Hadamard gate

\[ \left| b \right\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} \left| b \right\rangle \left| b \right\rangle + \frac{1}{\sqrt{2}} \left| b \right\rangle \left| \overline{b} \right\rangle \]

\[ \left| b \right\rangle \rightarrow \frac{\left( -1 \right)^b}{\sqrt{2}} \left| b \right\rangle \left| \overline{b} \right\rangle \left| \overline{b} \right\rangle \]