Overview of Lecture 16

• The GHZ “paradox”
• The Bell inequality and its violation:
  – Physicist’s perspective
  – Computer Scientist’s perspective
• The magic square game
preliminaries
Quantum information can apparently be used to substantially reduce *computation* costs for a number of interesting problems.

How does quantum information affect the *communication costs* of information processing tasks?

We explore this issue ...
Entangled states, such as $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$,

Entanglement and signaling

can be used to perform some intriguing feats, such as teleportation and superdense coding

But they cannot be used to “signal instantaneously”

Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)
Basic communication scenario

**Goal:** convey $n$ bits from Alice to Bob

$x_1x_2 \ldots x_n$

Alice

Resources

Bob

$x_1x_2 \ldots x_n$
Basic communication scenario

Bit communication:

Cost: \( n \)

Qubit communication:

Cost: \( n \) [Holevo’s Theorem, 1973]

Bit communication & prior entanglement:

Cost: \( n \) (can be deduced)

Qubit communication & prior entanglement:

Cost: \( n/2 \) superdense coding [Bennett & Wiesner, 1992]
nonlocality a là GHZ
GHZ scenario

[Greenberger, Horne, Zeilinger, 1980]

Input:
- $r$
- $s$
- $t$

Output:
- $a ← r$
- $b ← \neg s$
- $c ← 1$

Rules of the game:
1. It is promised that $r \oplus s \oplus t = 0$
2. No communication after inputs received
3. They **win** if $a \oplus b \oplus c = r \lor s \lor t$

<table>
<thead>
<tr>
<th>$rst$</th>
<th>$a \oplus b \oplus c$</th>
<th>$abc$</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0 ⚫</td>
<td>011</td>
</tr>
<tr>
<td>011</td>
<td>1 ⚫</td>
<td>001</td>
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<tr>
<td>101</td>
<td>1 ⚫</td>
<td>111</td>
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<tr>
<td>110</td>
<td>1 ☹</td>
<td>101</td>
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</tbody>
</table>
No perfect strategy for GHZ

Input: $r$
Output: $s$
Output: $t$

General deterministic strategy:

<table>
<thead>
<tr>
<th>$rst$</th>
<th>$a \oplus b \oplus c$</th>
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<tr>
<td>000</td>
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<td>1</td>
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<tr>
<td>101</td>
<td>1</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
</tr>
</tbody>
</table>

Winning conditions:

\[
\begin{align*}
& a_0 \oplus b_0 \oplus c_0 = 0 \\
& a_0 \oplus b_1 \oplus c_1 = 1 \\
& a_1 \oplus b_0 \oplus c_1 = 1 \\
& a_1 \oplus b_1 \oplus c_0 = 1 \\
\end{align*}
\]

Has no solution, thus no perfect strategy exists.
Input and output events can be \textit{space-like} separated: so signals at the speed of light are not fast enough for cheating.

What if Alice, Bob, and Carol \textit{still} keep on winning?
“GHZ Paradox” explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$

Alice’s strategy:
1. if $r = 1$ then apply $H$ to qubit
2. measure qubit and set $a$ to result

Bob’s & Carol’s strategies: similar

Case 1 ($rst = 000$): state is measured directly … 😒
“GHZ Paradox” explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$

Alice’s strategy:
1. if $r = 1$ then apply $H$ to qubit
2. measure qubit and set $a$ to result

Bob’s & Carol’s strategies: similar

Case 2 ($rst = 011$): new state $|001\rangle + |010\rangle - |100\rangle + |111\rangle$ 😞

Cases 3 & 4 ($rst = 101$ & $110$): similar by symmetry 😊
GHZ: conclusions

• For the GHZ game, any \textit{classical} team succeeds with probability at most $\frac{3}{4}$

• Allowing the players to communicate would enable them to succeed with probability 1

• Entanglement cannot be used to communicate

• Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1

• Thus, entanglement is a useful resource for the task of \textit{winning the GHZ game}
Bell’s Inequality and its violation part I
Bell’s Inequality and its violation
Part I: physicist’s view:

Can a quantum state have *pre-determined* outcomes for each possible measurement that can be applied to it?

qubit:

where the “manuscript” is something like this:

if \{|0\rangle,|1\rangle\} measurement then output 0

if \{|+,|\rangle\} measurement then output 1

if ... (etc)

(called *hidden variables*)

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]
Bell Inequality

Imagine a two-qubit system, where one of two measurements, called $M_0$ and $M_1$, will be applied to each qubit:

Define: $A_0 = (-1)^{a_0} A_1 = (-1)^{a_1} B_0 = (-1)^{b_0} B_1 = (-1)^{b_1}$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 (B_0 + B_1) + A_1 (B_0 - B_1) \leq 2$

one is $\pm 2$ and the other is 0

space-like separated, so no cross-coordination
Bell Inequality

\[ A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2 \] is called a **Bell Inequality**

**Question:** could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

**Answer 1:** *no, not directly*, because \( A_0, A_1, B_0, B_1 \) cannot all be measured (only one \( A_s B_t \) term can be measured)

**Answer 2:** *yes, indirectly*, by making many runs of this experiment: pick a random \( st \in \{00, 01, 10, 11\} \) and then measure with \( M_s \) and \( M_t \) to get the value of \( A_s B_t \)

The *average* of \( A_0 B_0, A_0 B_1, A_1 B_0, -A_1 B_1 \) should be \( \leq \frac{1}{2} \)

* also called CHSH Inequality
Violating the Bell Inequality

Two-qubit system in state

\(|\phi\rangle = |00\rangle - |11\rangle\)

Applying rotations \(\theta_A\) and \(\theta_B\) yields:

\[
\begin{align*}
\cos(\theta_A + \theta_B) (|00\rangle - |11\rangle) + 
\sin(\theta_A + \theta_B) (|01\rangle + |10\rangle)
\end{align*}
\]

Define

\(M_0\): rotate by \(-\pi/16\) then measure

\(M_1\): rotate by \(+3\pi/16\) then measure

Then \(A_0B_0, A_0B_1, A_1B_0, -A_1B_1\) all have expected value \(\frac{1}{2}\sqrt{2}\), which contradicts the upper bound of \(\frac{1}{2}\)
Bell Inequality violation: summary

Assuming that quantum systems are governed by *local hidden variables* leads to the Bell inequality

\[ A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2 \]

But this is *violated* in the case of Bell states (by a factor of \( \sqrt{2} \))

Therefore, no such hidden variables exist

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted.
Bell’s Inequality and its violation part II
Bell’s Inequality and its violation

Part II: computer scientist’s view:

input:

```
   s
   |
   v
 a
```

output:

```
   t
   v
 b
```

Rules: 1. No communication after inputs received
2. They **win** if \( a \oplus b = s \land t \)

With classical resources, \( \Pr[a \oplus b = s \land t] \leq 0.75 \)

But, with prior entanglement state \(|00\rangle - |11\rangle\),
\( \Pr[a \oplus b = s \land t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = 0.853\ldots \)
The quantum strategy

• Alice and Bob start with entanglement

$$|\phi\rangle = |00\rangle - |11\rangle$$

• **Alice:** if $s = 0$ then rotate by $\theta_A = -\pi/16$
  else rotate by $\theta_A = +3\pi/16$ and measure

• **Bob:** if $t = 0$ then rotate by $\theta_B = -\pi/16$
  else rotate by $\theta_B = +3\pi/16$ and measure

$$\cos(\theta_A - \theta_B) (|00\rangle - |11\rangle) + \sin(\theta_A - \theta_B) (|01\rangle + |10\rangle)$$

Success probability:

$$\Pr[a \oplus b = s \land t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = 0.853...$$
Nonlocality in operational terms

- Information processing task
- Classically, communication is *needed*
- Quantum entanglement
Magic square game

**Problem:** fill in the matrix with bits such that each row has even parity and each column has odd parity

$$
\begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33} \\
\end{array}
\begin{array}{ccc}
    \text{even} & \text{even} & \text{even} \\
    \text{odd} & \text{odd} & \text{odd} \\
\end{array}
$$

**Game:** ask Alice to fill in one row and Bob to fill in one column

They *win* iff parities are correct and bits agree at intersection

**Success probabilities:** $\frac{8}{9}$ classical and 1 quantum

[Aravind, 2002] (details omitted here)
preview of communication complexity
Classical Communication Complexity

[Yao, 1979]

\[ f(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases} \]

E.g. equality function: \( f(x,y) = 1 \) if \( x = y \), and \( 0 \) if \( x \neq y \)

Any **deterministic** protocol requires \( n \) bits communication

**Probabilistic** protocols can solve with only \( O(\log(n/\varepsilon)) \) bits communication (error probability \( \varepsilon \))
Quantum Communication Complexity

Qubit communication

Prior entanglement