Some Polynomial and Toeplitz Matrix Computations

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1. Introduction to Parts 1 and 2

In the vast bibliography on the evaluation of complex polynomial zeros, only a few works specifically address the complexity issue, see [22, 28] for surveys. The complexity estimates in those papers have been obtained via several different algorithms, relying on Newton's iteration and on the various techniques of computing power sums and contour integrals. Three basic approaches and techniques have also been manipulated in different ways and combined with each other and with some geometrical constructions for search and exclusion in complex domains, due to Weyl, Lehmer, Range. In particular Smale and then Shub and Smale, proved that Newton's method is highly effective in the average case, see [28], while Turing's power sum method of 1968 and its recent extensions turned out to be good and reliable in the worst case, leading to the record worst case complexity bounds of $O(n \log n \log \log n \log (1/e))$ arithmetic operations for approximating all the complex zeros $x_1, \ldots, x_n$ of an $n$-th degree polynomial $p(x)$ with relative errors $< e$ and of $O(n \log n \log (1/e))$ for similarly approximating a single zero, see [20, 22, 28]. Later on Renegar in [25] approximated all the zeros of $p(x)$ with absolute errors $< e$ using $O(n \log n \log (1/e))$ arithmetic operations, provided that $p(x)$ has been scaled so as to make all its zeros lie in a unit circle. This improved the above estimate of [22] for approximating all the zeros of $p(x)$ (although the improvement is minor in the cases where $|\log \epsilon| = O(n \log n)$). Our new algorithms of this paper improve the estimates of [22] and [25] for the arithmetic computational complexity of computing all the zeros of $p(x)$ roughly by a factor of $n^2 \log n$, establishing a new asymptotic record bound. Furthermore, computing a single zero of a polynomial whose all zeros are real, we reach the upper bound $O(n \log n \log n \log (1/e))$, which substantially decreases the bound of [22] in the cases of larger $\epsilon$ and which decreases the respective bound of [25] roughly $n^2$ times. This brings us quite close to the known lower bound $n^2 \log n \log (1/e)$, (At least $n^2$ arithmetic operations are needed already in order to process a input coefficients of $p(x)$; the lower bound $O(n \log (1/e))$ follows from the stronger results of [16], rediscovered in a weaker form in [25]). Besides, our overhead constants are small; in particular they substantially improve over those of [25]. It is interesting to compare the techniques of this paper with those of [25]. As this is emphatically proclaimed in the summary and throughout the paper [22], the algorithm of [25] "is built around Newton's method and Schur-Cohn algorithm". In our paper we share Weyl's geometric construction with [25], but we rely on completely distinct algebraic and analytical tools, that is, on numerical integration (successfully replacing Newton's iterations) and power sum computations, particularly on Turing's proximity test and on its extension in [27] to root radius computation: nowhere we have to apply Newton's method or Schur-Cohn test. (In April 1987 a draft of this paper was sent to J. Renegar, he then used some of our observations in order to improve his original (but not our new) complexity bounds for the price of abandoning only Schur-Cohn's test, but not Newton's method.}

Our new algorithms are also effective for parallel evaluation of a single zero and of all the zeros of \( p(x) \), see Table 1.1 below. For parallel computations we assume the customary machine model, where in each step each processor performs at most one arithmetic operation, that is, \( +, -, \cdot, \div \), or the evaluation of an \( N \)-th root of a positive number for a natural \( N \). (We will also include here the comparison of two real numbers.) We estimate the number of processors up to within constant factors for we may always slow down the computations \( K \) times and use \( K \) times fewer processors (but \( \geq 1 \)).

### Table 1.1. Arithmetic complexity of approximating polynomial zeros in unit circle with absolute errors \( \leq 2^{-b} \).

<table>
<thead>
<tr>
<th></th>
<th>Sequential time</th>
<th>Parallel time, Processors</th>
</tr>
</thead>
<tbody>
<tr>
<td>All zeros</td>
<td>( O(n^2 (\log^2 n + \log b)) )</td>
<td>( O(n (\log^2 n + \log b, n) )</td>
</tr>
<tr>
<td>Single zero (general case)</td>
<td>( O(n \log (n+\log b)) )</td>
<td>( O(\log (n (\log^2 n + \log b), n+n^2/(\log^2 n \log b)) )</td>
</tr>
<tr>
<td>Single zero (all zeros are real)</td>
<td>( O(n \log (n^2 + \log b)) )</td>
<td>( O((\log^2 n + \log b, n) )</td>
</tr>
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Table 1.1 shows in particular that the parallel evaluation of a single zero of \( p(x) \) requires only polylogarithmic parallel time and less than \( n^2 \) processors, so we add a new problem to NC, [2, 8]. In Section 11 we show that \( O(bn) \) bit-precision of computations suffices in order to support the arithmetic complexity estimates of Table 1.1 where \( b = q + m, \epsilon = 2^{-9} \), and the coefficients \( c_0, c_1, \ldots, c_n \) of \( p(x) \) are integers such that \( \epsilon | c_1 | < 2^m \).

The study of perturbation of polynomial zeros due to variation of coefficients, see [19], pp. 74-77; [18, 27], suggests that the precision of computations must be at least of an order of \( bn \) in the worst case, but on the other hand, computing with the precision \( O(b) \) usually suffices in practice in order to evaluate the non-clustered complex zeros of many input polynomials \( p(x) \) with errors \( \leq 2^{-b} \) relative to the coefficient size and also to factor an \( arbitrary \) univariate polynomial numerically in the complex domain.

Since \( O(h \log b \log \log b) \) Boolean operations or \( O(\log^2 h) \) Boolean parallel steps, \( h \log b \log \log b \) processors suffice for an arithmetic operation with integers modulo \( 2^h \), \([1, 5, 17, 20]\), and since in our case \( h = O(bn) \), we may immediately extend our estimates of Table 1.1 to the case of the Boolean circuit complexity model. In particular we arrive (within a logarithmic factor) at the Boolean sequential time bound on the complexity of computing all the zeros of \( p(x) \) stated in [27], Sect. 19 but only partly proven so far. (Schönhage [27] needs to use \( n^2 \) arithmetic operations already in his \( n \) stages of numerical integration, which exceeds the bound of Table 1.1 about \( n \) times, but he computes with lower precision in those stages and with the precision \( O(bn) \) in other stages.) His complete proof of the Boolean complexity bound seems to be very much involved and has not appeared yet. There seems to be two difficulties with that proof. One difficulty is due to the intention of Schönhage to supply various techniques for the study of the asymptotic complexity of arithmetic computations with multiple precision. That study is important theoretically and may lead to the results of practical value. Computing polynomial zeros is certainly a good example where multiple precision is required. Another difficulty is due to the approach of [27], where major efforts are spent to separate the contours lines (used for numerical integration) from polynomial zeros. Those efforts, supported by elaborate and intricate error analysis, lead to quite favorable asymptotic estimates for the Boolean complexity of computing polynomial zeros (provided that the proofs will be completed), but the overhead constants substantially grow. (Those constants are hidden in the \( O \) notation of the asymptotic estimates of [27].)

In contrast to that, in this paper the zeros of \( p(x) \) are included into regions of regular shapes (circles or squares) and are evaluated using Weyl’s construction, which naturally subdivides the computations into isolated and self-clearing stages. All this simplifies the complexity analysis and enables us to decrease the overhead constants, making our algorithms good candidates for practical implementation. Furthermore the arithmetic complexity bounds of the first two lines of Table 1.1 seem to be overly pessimistic in the case of many polynomials \( p(x) \). While deducing our worst case estimates, we pessimistically assume that all our recursive subdivisions of the set of zeros of \( p(x) \) are highly unbalanced, while again such a systematic disbalance is certainly a rather exceptional case in practice.

Further extensions and applications of our results include, in particular, a) the evaluation of the eigenvalues of a matrix via the evaluation of its characteristic polynomial, see [22], b) factorization of a polynomial in the complex domain via the same algorithms for computing all the zeros (with more favorable Boolean complexity bounds for factorization), compare [27], Theorems 2.1 and 19.2, and c) computing the greatest common divisor (gcd) of two or several polynomials. Numerical treatment of the latter problem can be reduced to numerical evaluation of all the zeros of the input polynomials.

That reduction, however, cannot be suggested for parallel computation, for the evaluation of polynomial zeros requires superlinear parallel time versus \( O(\log^2 n) \) time for the polynomial gcd computations, [8]. That parallel time bound for the gcd, however, is supported by using a quite high processor bound. This leads us to our second major subject. We treat that subject (more broadly) as parallel computations with Toeplitz matrices \( T = \{ t_{ij} \}, t_{ij} = t_{i+j}, i,j = 0,1,\ldots,n-1 \) (which includes the gec computations as an important but very particular case). Fast and processor efficient evaluation of the inverse \( T^{-1} \), the determinant \( \det(T) \), and the characteristic polynomial \( \det(\lambda I - T) \) of \( T \) should have important applications to time series analysis, image processing, control theory, statistics, solution of integral and partial differential equations, Padé approximations, rational interpolation, partial fraction decomposition, computations with polynomials, and with matrix polynomials, see [7, 8, 30, 31]. Our main result is a parallel algorithm for the above evaluations for an \( n \times n \) Toeplitz matrix \( T \). That algorithm runs in \( O(\log^2 n) \) parallel arithmetic time and requires only \( O(n^2) \) processors. The precision of computation of \( O(a \log(n||T||) \) binary bits suffices, where \( ||T|| = \max_{ij} |t_{ij}| \). Our arithmetic complexity bounds and our precision bounds immediately lead to a respective substantial improvement of the known bounds on the parallel Boolean circuit complexity.

For comparison the previous parallel Toeplitz algorithms either run in linear arithmetic time (of order \( n \)) or otherwise required at best \( M(n) \) processors and the precision of an order of \( n \log(n||T||) \) binary bits, the same as for the general matrix computations, see [19-21]. Here \( M(n) = O(n^2) \) denotes the sequential time required for \( n \times n \) matrix multiplication, currently in theory \( \omega \leq 2.375 \) with huge overhead constants, and in practical computations \( \omega = 3 \) with small overhead. Thus our improvement means saving an order of \( n \) processors (if we refer to the practical bound \( \omega = 3 \)). Furthermore our parallel algorithm for
computing \( \det(A-T) \) has sequential time \( O(n^{2}\log^{2}n) \) arithmetic operations, which stays on the level of the current best sequential time estimates for that computational problem. The current best arithmetic sequential time for the inversion of \( T \), however, is about \( n \) times less than the sequential time of our algorithm, \( O(n \log^{2}n) \), see [4, 7]. Furthermore, to invert a Toeplitz matrix \( T \), we need to compute the coefficients of its characteristic polynomial, which may be fairly large, while some of the cited sequential algorithms for numerical inversion of \( T \) involve large numbers only where \( T \) is ill-conditioned, see [8].

On the other hand, many important computations (such as rational interpolation and computing the greatest common divisor of two or several polynomials) are reduced to solving ill-conditioned Toeplitz linear systems, and then even the sequential version of our algorithms can be made competitive with, and even slightly superior to the current best algorithms, via the techniques used in [20, 21]. Namely, we may assume that the entries of the input Toeplitz matrix \( T \) are integers (the real or complex binary entries can be truncated to finite precision binary numbers, and then the matrix \( T \) can be scaled to turn those binary numbers into integers) and evaluate \( \det(T) \) and \( \text{adj}(T) = T^{-1}\det(T) \) modulo a prime power \( p^k \), that is, we compute with \( \lfloor \log_{p} p^k \rfloor \) binary bits. We choose \( p \) and \( k \) such that \( \log_{p} p^k \) (slightly) exceeds the number of bits required to represent each output value \( \lfloor \log_{p} \det(T) \rfloor \) + 1 bits would suffice for any \( T \) (for some \( T \) even fewer bits suffice). Then we easily recover all the exact integer output values (that is, \( \det(T) \) and the entries of \( \text{adj}(T) \)) from their values mod \( p \) and arrive at the desired exact solution, including also \( T^{-1} = \text{adj}(T)/\det(T) \), compare [20, 21]. The precision of those computations (measured by the number of binary bits used in order to represent the operands) is within one binary bit from the number of bits already required in order to represent the absolutely largest output value (assuming that we compute the outputs exactly, which is a reasonable assumption in the case of ill-conditioned linear systems). Surprisingly though, we may compute with only \( \lfloor \log_{p} p^k \rfloor \) binary bits throughout, except for some stages involving \( O(n \log^{2}n) \) arithmetic operations. We will do this via the Newton-Hensel lifting algorithm and will arrive at a sequential algorithm for the exact inversion of Toeplitz matrices, whose overall asymptotic Boolean cost is even lower than the cost of the current best algorithms, see [23].

We hope that the techniques that we used are of some independent interest. Our algorithm relies on the reduction of the auxiliary matrix powering (which destroys the Toeplitz structure) to matrix inversion (which does not). That reduction is akin to the variable diagonal technique of [20, 21]. For matrix inversion we use Newton's iterations along the line of [19], but we had to modify those iterations in order to preserve the lower displacement rank of Toeplitz matrices, which characterizes their structure, compare [12]. Also we exploit the Czamay-LeVerrier construction of [10], with an improvement in the stage of solving a system of Newton's identities.

Our present main objective is to demonstrate the power of the concepts and techniques used in our algorithm and to deduce the desired estimates for parallel complexity. With more work and further elaboration our method should lead to more effective parallel algorithms more suitable for implementation. We can see and foresee many further applications of our results to computations with Toeplitz and other structured matrices (such as Hankel, block Toeplitz and block Hankel matrices, matrices and block matrices having small displacement ranks). In particular, the greatest common divisor of \( m \) polynomials of degree \( n \) with integer coefficients ranging from \( -t \) to \( t \) can be computed using \( O(t^{2}n \log^{2}(m+n)) \) parallel arithmetic steps, \( m^{2}n \) processors, and the bit-precision of computations \( O(n \log(mnt)) \). This immediately leads to improvements of the known parallel algorithms for several other polynomial and rational computations, such as rational (Cauchy and Hermite) interpolation, Padé approximation, computing elementary symmetric functions, Taylor expansion, Chinese remainder algorithm, partial fraction decomposition, square free decomposition of a polynomial, computing the least common multiple of two or several polynomials, computing polynomial resultants, and so on, compare [7, 30, 31].

Part 1: Approximate Evaluation of Polynomial Zeros

2. Definitions and An Auxiliary Algorithm

Definition 2.1. compare [6]. \( O_{A}(t,p) \) means \( t \) parallel arithmetic steps and \( p \) processors. \( O_{A}(T) = O_{A}(T, 1) \) denotes the sequential arithmetic time, that is, the number of arithmetic operations used. Replacing arithmetic operations by Boolean ones we arrive at the Boolean model of computations with the notation \( O_{B}(t,p) \), \( O_{B}(T) \).

Let a monic polynomial \( p(x) \) of degree \( n \) be fixed,

\[
p(x) = \sum_{i=0}^{n} c_{i}x^{i} = \prod_{j=1}^{s} (x-x_{j}), \quad c_{n} = 1. \tag{2.1}
\]

Definition 2.2. \( D = D(X,R) \) denotes the disc of radius \( R \) with center \( X \) on the complex plane; \( S = S(X,R) \) denotes the square with vertices \( X + R, \ X - R, \ X + R
eq \overline{r}, \ X - R \pm \sqrt{1}; \ A = A(X,R,r), \ R > r, \) denotes the annulus \( D(X,R) - D(X,r) \). We will write \( \rho(S) = \rho(D) = R, \rho(A) = r-r. \)

Remark 2.1. Hereafter in different sections each of the characters \( R, \ r, \ x, \ y, X, Y, R_{x}, R_{y}, Y_{x}, Y_{y}, \) and so on may denote distinct values; all the rectangles and squares in complex domains have their sides parallel to the coordinate axes.

Definition 2.3. \( x(U) \) denote the set of all the zeros of \( p(x) \) in a complex domain \( U \). Two complex domains \( U, V \) are called equivalent if \( x(U) = x(V) \). Transformation of a domain \( U \) into its equivalent subdomain is called shrinking or contraction. If \( U \) denotes a window, a disc, or an annulus, we define its rigidity ratio \( r.r(U), \) and its isolation ratio, i.r.(U), as follows,

\[
r.r.(U) = \min(\rho(U)/\rho(U)), \quad i.r.(U) = \min(\rho(U)/\rho(U)),
\]

Here the minimization is over all the domains \( U \) and \( U^{*} \) equivalent to \( U \) and such that \( U \subseteq U \) and \( U \subseteq U^{*} \).

Definition 2.4. \( d(U) = \max \{ x_{k} - x_{l} \} \) for a complex domain \( U \). Here the maximum is over all the pairs of zeros of \( p(x) \) in \( U \).

Proposition 2.1. \( r.r.(S) \geq d(S)/(2\sqrt{\rho(S)}) \) for a square \( S; \ r.r.(D) \geq d(D)/(2\rho(D)) \) for a disc \( D \).

Definition 2.5. A complex point \( X \) is called an isolated \( \epsilon \)-approximation to a zero \( x_{i} \) of \( p(x) \) if the disc \( D(X,\epsilon) \) contains \( x_{i} \) and has isolation ratio at least \( 1 + 1/n \).

Definition 2.6. The number of zeros of a polynomial \( p(x) \) in a complex domain \( U \), counted with their multiplicities, is called the index of \( p(x) \) in \( U \) and denoted i.i.(p(x),U).

Definition 2.7. The distances \( r_{0}(X) \geq r_{1}(X) \geq \ldots \geq r_{g}(X) \) from a point \( X \) to the \( n \) zeros of \( p(x) \) are called the root radii of \( p(x) \) at \( X; \ r_{g}(X) \) is called the \( g \)-th root radius of \( p(x) \) at \( X \).
Definition 2.8. For fixed positive $R$ and $\epsilon$,

$$b = \log_2 \left( \frac{R}{\epsilon} \right).$$

(2.2)

Algorithm 2.1. Supersession of a square about a given set of points. Input. A (finite) set $H$ of complex points. Computation. Compute the maximum $M$ and the minimum $m$ in the set of the real parts of the points of $H$. Output $(M+m)/2$ and $(M-m)/2$. Then repeat the same computations for the set of the imaginary parts of the points of $H$.

Proposition 2.2. Compare Remark 2.1. The two half-sums in the outputs of Algorithm 2.1 equal the real and imaginary parts of the center of the minimum rectangle containing $H$. The two half-differences equal the half-lengths of the sides of that rectangle. The center $x$ and the half-length $r$ of the longer side define a square $S(x,r)$ containing the set $H$. Moreover $r \leq p(S)$ for any square $S$ containing $H$.

In the sequel we will use the known algorithms for some basis operations with polynomials (such as their multiplication and division with a remainder, discrete Fourier transform (DFT), scaling and shift of the variable) and for solving a triangular Toeplitz system of equations; the arithmetic cost of those computations is $O_N(\log n,n)$, see [1, 5, 17, 22].

3. Turan's Proximity Test

Algorithm 3.1 (Turan's proximity test).

Input. A degree $n$ polynomial $p(x)$ of (2.1) and a complex number $X$ that is not a zero of $p(x)$.

Stage 1. Choose a natural $N$ and compute the values of the (shifted inverse) power sums,

$$s_{1,N} = \sum_{j=1}^{\infty} \frac{y_j}{j^{1/N}}, \quad y_j = 1/(x-x_j), \quad j=1,2,\ldots,n.$$  

Stage 2. Compute and output $r = \max_{j=1,\ldots,n} |s_{1,N}/n|^{1/(n+1)}$ and $r' = R^{-1/N}$.

Theorem 3.1. [29], p. 299. For the output $r$ of Algorithm 3.1,

$$1 \leq \min_{x \in D} |X - x| / r \leq R^{1/N},$$

that is, if $p(x,D) = 0$, $r := R^{1/N}$ where $D = D(X,r)$, see Definitions 2.3, 2.6.

For our purpose it will suffice if $N = 32$; then $1.05185 < R^{1/N} < 1.05182$. [31]

[29] performs Stage I for $N = 2^4$ as follows.

Subalgorithm 3.1.1

Stage a). Shift the variable $y = X - X$ and compute the coefficients of the $n$-th degree polynomial $q(y)$ with the zeros $y_j = 1/(x-x_j), j = 1,\ldots,n$ such that

$$p(x) = p(X + y) = \sum_{i=0}^{n} c_i(x) y^i,$$

$$q(y) = y^n p(X + 1/y) = \sum_{i=0}^{n} c_i(x) y^{-i} = c_n(x) \prod_{j=1}^{n} (y - y_j).$$

Stage b). Let $q_1(y) = q(y)/c_n(X)$ and successively compute the coefficients of the polynomials

$$q_{i+1}(y) = q_i(\sqrt{y}) q_i(-\sqrt{y}), i = 0,1,\ldots,n-1.$$

4. Computing the Number of Polynomial Zeros in an Isolated Disk

The well known winding number algorithms (whose cost is $O_N(\log n,n)$, [14], pp. 239-241; [25]), compute the index $i(p(x),D)$ of $p(x)$ in a disc $D(D(x,r))$ provided that all the zeros of $p(x)$ lie far enough from the boundary of the disk or of a fixed disc $D(x,r')$ equivalent to $D$. [25] proves that the bound $i(r,D) \geq 9$ already suffices to assure the latter assumption, see Remark 2.2 in [24] for an alternative solution having the same cost. Let us reduce the case of arbitrary disc $D$ with $i(r,D) > 1$ to the case $i(r,D) \geq 9$.

Algorithm 4.1. Inputs. Complex $X$, positive $r$ and $\nu$, and polynomial $p(x)$ of (2.1) such that $i(r,D(x,r)) \geq 1+\nu > 1$, compare Definition 2.3.

Stage 1. Compute the coefficients of the monic polynomial $q_0(y) = p(x+ry) / r^2$, $i(p(x),D(X,r)) = i(q_0(y),D(0,1))$.

Stage 2. Apply $\nu$ iterations (3.3) where

$$h = \left[ \log(\log(9/\log(1+\nu))) \right] = O(\log(1+\nu)) \quad \nu \to 0.$$  

(4.1)

Stage 1 of the algorithm transforms the discs $D(X,r)$ and $D(X(1+r^2))$ into the discs $D(0,1)$ and $D(0,1+\nu)$, respectively. Stage 2 transforms the disc $D(0,1)$ into itself and the disc $D(0,1+\nu)$ into the disc $D(0,1+\nu)^2$, where $\nu^2 \geq 9$ due to (4.1), so the isolation ratio of $D(0,1)$ with respect to $q_0(y)$ is at least 9. Therefore, we may apply a winding number algorithm and compute the index $i(q_0(y),D(0,1)) = i(p(x),D(X,r))$.

Proposition 4.1. Let $i(r,D) \geq 1+\nu > 1$ for a disc $D$. Then the index $i(p(x),D)$ can be computed for the cost $O_n(h \log n,n)$ where $h$ is defined by (4.1).

Corollary 4.1. Given an isolated $c$-approximation $X$ to a zero of $p(x)$, the index $i(p(x),D)$ in the disc $D(X,c)$ can be computed for the cost $O_n(l \log n)$.

Proof. Apply Proposition 4.1 with $c = c_i, i = 1/n$, Q.E.D.

Remark 4.1. The results of this section would not change if we set $q(y) = p(x+y)$ in Stage 1 of Algorithm 4.1.

5. Turan-Weyl's Exclusion Algorithm

Algorithm 5.1 (Turan-Weyl).
Stage 1. Let $g$ and $h$ take the values $-1$ and $1$ and let

$$Y(g,h) = Y + (g+h\sqrt{-1})R$$

denote the four vertices of the square $S(Y,R)$. Fix $N$, apply Algorithm 3.1 to the polynomial $p(x)$ at the four points $Y(g,h)$, and denote the output values $r(g,h), r'(g,h)$ (rather than $r, r'$).

Proposition 5.1. Iteration $j$ of Algorithm 5.1 has cost $O_A(n \log n/k)$ and defines at most $4k$ suspect squares with side length $R/2^J$.

Proposition 5.2. Let $c$ be a constant positive, $b = \log(n/R)$, compare (2.2). Then the center $X_i$ of every suspect square output by iteration $j = \lfloor b + \log_2 n \rfloor + 5$ of Algorithm 5.1 is an isolated $\epsilon_i$-approximation to a zero of $p(x)$ for $\epsilon_i \leq \epsilon$ for all the suspect squares can be computed for the cost $O_A(b \log n/k)$, so is an isolated $\epsilon_i$-approximation to all the $k$ zeros of $p(x)$ in $S(X,R)$ where $\epsilon_i \leq \epsilon$ for all $i$ can be computed for the cost $O_A(b \log n/k)$.

Proposition 5.3. Each output suspect square $S(X_i,R)$ has center $X_i$ approximating to a zero of $p(x)$ within $\sqrt{2} \sqrt{R}$, and

$$r < c/(12n+1) \sqrt{2}. \quad (5.1)$$

Also other required properties immediately follow, except that it remains to prove the $\epsilon_i$-isolation of $X_i$ with $\epsilon_i \leq \epsilon$ and to estimate $\epsilon_i$ for each center $X_i$. Define

$$\epsilon_i = r \sqrt{2}, \quad r_{i+1} = r_{i} + 3\theta = (3i+1)r \quad \text{for } i = 0,1,2,\ldots \quad (5.2)$$

Successively (for $i=0,1,\ldots$) check if any suspect square does not lie in the disk $D(X_i,r_i)$, but lies in or intersects the disk $D(X_i,r_{i+1})$ and therefore lies inside the disk $D(X_i,r_i)$. Since there are at most $4k$ suspect squares, checking step $i$ will give answer "no" for some $i = w(s) \leq 4$. Then

$$r_i = (3i+1)r \leq (12k+1)r < c, \quad \text{and } i = i(D(X_i,r_i)) \geq 1/1+i/a, \quad \text{so } X_i \text{ is a desired isolated } \epsilon_i-\text{approximation to a zero of } p(x) \text{ for } \epsilon_i = r_i. \quad \text{For every fixed } s \text{ perform all the } O(k) \text{ checking steps in parallel. } \text{Q.E.D.}$$

Remark 5.1. The index of $p(x)$ in the $\epsilon_i$-neighborhood of each isolated $\epsilon_i$-approximation to a zero of $p(x)$ is computed by Algorithm 4.1.

5. How to Contract a Square Region

For the cost $O_A(n \log n)$, the next algorithm will contract a square $S(Y,R)$ having isolation ratio $\geq 3$ into its subsquare $S(Z,R)$ such that either

$$r < 0.8R \quad (6.1)$$

or

$$r \cdot r(S(Z,R)) > 0.1 \text{ and } d(S(Z,R)) > 0.3R > 0.3r. \quad (6.2)$$

compare Definitions 2.3 and 2.4.

Subalgorithm 6.1. Inputs. Polynomial $p(x)$ of $(2.1)$, complex $Y$, and positive $R$, such that $i \cdot r(S(Y,R)) \geq 3$.

Next prove for $h=1$ that

$$d(h) = 2\sqrt{2R} - (2\sqrt{2} - d(1,h))/\sqrt{2} \quad (6.3)$$

where $d(1,h)$ denotes $\text{dist}(D(Y(1,h),r(1,h))) = |Y(1,h) - Y(1,h)| = r(1,h) - r(1,h)$.

Let $E_1, E_2, E_3, E_4$ denote four successive points of the diagonal of the square $S(Y,R)$ passing through $Y(1,1)$ and $Y(1,1)$, namely, the four intersection points of that diagonal with the four lines: $L(1,1), L(1,1), L(1,1), L(1,1)$. Thus $A_1$ and $B_1$ denote the two intersection points of $L(1,1)$ with the two sides of the square $S(Y,R)$ for $g=1$ and $g=1$. Then

$$d(1) = |E_1 - E_2|, \quad d(1) = |E_1 - E_3|, \quad |Y(1,h) - E_1| = |Y(1,h) - E_2| = |E_1 - E_3| = \sqrt{2}$$

for $g=1$ and $g=1$, so $2\sqrt{2} - d(1) = |Y(1,1) - Y(1,1)| - d(1) = \sum |Y(1,h) - F_q| = \sum |Y(1,h) - F_q|/\sqrt{2} = \varepsilon$.\]
(2√2R - d(1,1))/√2. This proves (6.8) for h=1; similarly (6.6) can be proven for h=1. Recall (6.5) and deduce that 
(2√2R-d) ≤ 4d-2r, where d = max{d(1,1),d(1,-1)}. Now (6.4) follows because 2√2R-d = (2√2R-d)1/2.N. Q.E.D.

Corollary 6.1. If R ≥ r ≥ 0.8R then
\[ d(S(Z,r)) ≥ \text{dist}(D_1,D_2) = d^* > (2√2-2.4)|S|^{1/3}/R. \]  
(6.7)

Otherwise (6.1) holds.

Next let N=32, apply (3.1) and Proposition 2.1, and deduce (6.2) from (6.7). Since i.e.(S(Z,r)) ≥ 3, Subalgorithm 6.1 can be applied again (with the inputs p(x), Z, r, k). Continue that process recursively (call it Algorithm 6.1) and arrive at

Proposition 6.2. For a positive \( E \), a square \( S(Y,R) \) having isolation ratio \( R ≥ 3 \) can be contracted for the cost \( O_n(\log n,n) \), \( g(\log(R/E)) \), into a square \( S(Z,s) \) such that \( r < E \) or else the relations (6.2) are satisfied.

7. Accelerated Shrinking of an Isolated Square

In this section we will rapidly contract a square \( S(Y,R) \) having isolation ratio at least \( 3 \) either i) into a disc of radius \( \epsilon \) (so its center approaches to all the zeros of \( p(x) \) lying in \( S(Y,R) \) with errors \( \leq \epsilon \)) or ii) into a square output by Subalgorithm 6.1 and satisfying the relations (6.2). At first we contract the square \( S(Y,R) \) into a disc having isolation ratio, say \( \geq 10n \), applying Proposition 6.2. Then we will need some auxiliary results. Denote

\[ M = \sum_{k=1}^{k} x_{k}(k)/k. \]  
(7.1)

If \( k=n \), then the value \( M = c_{-1}/n \) is immediately available, see (2.1). In the next section we will prove the following proposition.

Proposition 7.1. For two positive constants \( \epsilon \) and \( \epsilon \), a natural \( k \), and a disc \( D \) such that \( i.e.(D) ≥ (1+\nu)^2, \nu > 0 \), \( i.e.(p(x)) = k \), the center of gravity \( M = M(D) \) of the \( k \) zeros of \( p(x) \) in \( D \) can be approximated by a point \( M' \) in \( D \) with absolute error \( \leq (1+\nu)^2\epsilon r \) for the cost \( O_n(\log n,n) \).

In the sequel we will also use the following proposition.

Proposition 7.2. Let \( p(x) = k \) for a disc \( D \) and let \( M \) denote the center of gravity of the \( k \) zeros of \( p(x) \) lying in \( D \). Then \( d(D) ≥ r_{k,M}(M)k/(k-1) \), compare Definitions 2.4, 2.7.

Proof. Let \( x_{k}(1),...,x_{k}(k) \) denote the \( k \) zeros of \( p(x) \) lying in the disc \( D \) and let \( M = \sum_{k=1}^{k} x_{k}(k)/k \). Without loss of generality, assume that \( M = 0 \); \( x_{k}(k) \) is negative. Then

\[ \sum_{k=1}^{k} k \text{ Re } x_{k}(k) = 0. \]  
Therefore \( \sum_{k=1}^{k} k \text{ Re } x_{k}(k) = r_{k,M}(M), \) 
\[ \max_{k=1}^{k} k \text{ Re } x_{k}(k) ≥ r_{k,M}(M)/(k-1), \]  
so \( d(D) ≥ r_{k,M}(M)/k(k-1). \) Q.E.D.

Corollary 7.1. Let under the assumptions of Proposition 7.2, \( |M - M'| ≤ r_{k,M}(M) = a \) for some complex \( M' \), and let \( D' \) denote the disc \( D(M',(1+\alpha)r_{k,M}(M)) \). Then \( d(D') ≤ d(D,M,r_{k,M}(M)), \) \( r_{k,M}(M') ≤ (1+\alpha)r_{k,M}(M), \) \( d(D') ≥ d(D) ≥ r_{k,M}(M)/((1+\alpha)k(k-1)). \) Q.E.D.

The next algorithm rapidly contracts a disc, as desired.

Algorithm 7.1.

Inputs. Polynomial \( p(x) \) of (2.1), complex \( X \), natural \( k \), and positive \( r \) and \( \nu \) such that
\[ i.e.(p(x)) = k, r.i.e.(D(X,r)) = (1+\nu)^2, \nu > 0. \]  
\[ \max_{x \in X} (4,256n^2)^{1/3}/2. \]  
(7.2)

Let \( M \) denote the unknown center of gravity of the \( k \) zeros of \( p(x) \) lying in \( D(X,r) \).

Initialization. \( j=0, M_0=X, r_0=r, \nu_0=\nu, D_0=D(M_0,r_0) \).

Stage \( j \), \( j=0,1, 2, \ldots \), \( J=1 \). At first compute (for the cost \( O_n(\log n,n) \)) an approximation \( M_{j+1} \in D(M_j,r_j) \) to \( M \) with absolute error bound
\[ \epsilon_{j+1} = (1+\nu)^2r_j \]  
(7.3)

(apply Proposition 7.1 where \( c=1, c_0=\epsilon, M=M_j, r=r_j, \nu=\nu_j \), compare also Proposition 7.3 below). Then compute an approximation \( r_{j+1} \) to the \( (n-k) \)-th root radius \( r_{k,M}(M_{j+1}) \) of \( p(x) \) at \( M_{j+1} \) with relative error at most \( 8n \). (The cost is again \( O_n(\log n,n) \), see [24].) Denote
\[ r_{j+1} = 8n, r_{j+1} = D(M_{j+1},r_{j+1}), (1+\nu_j) = (1+\nu)^2. \]  
(7.4)

If \( r_{j+1} < \epsilon \) set \( J=j+1, \) output \( M_j \) and \( r_j \) and end the computations. If
\[ \epsilon \leq r_{j+1} < 256n^2(1+\nu)^2r_{j+1} \]  
(7.5)

enter Stage \( j+1 \). Otherwise set \( J=j+1 \) and apply Algorithm 6.1 to the square \( S(M_j,r_{j+1}) \) superscribing the disc \( D_j \) (until the relations (6.2) are satisfied).

Let us analyze Algorithm 7.1. At first consider the case where (7.5) holds for all \( j \). Then recursively apply the last equation of (7.4) and obtain that \((1+\nu_j)^2 = (1+\nu)^2 / j, j=1,2,\ldots \)

Substitute this into (7.5), apply the resulting relation recursively, and deduce that in this case
\[ \log(\epsilon/\epsilon_j) > 4\log(1+\nu) \]
\[ r_j < \epsilon \] if \( J = \lceil \log(\log(\epsilon/\epsilon_j) - 2 - \log(\log(1+\nu)/\log n)) \rceil \).

We need, however, the following result in order to apply Proposition 7.1 when we deduce (7.3).

Proposition 7.3. If (7.5) holds for \( j=0,1,2,\ldots \), then
\[ (1+\nu_j)^2 ≤ i.e.(D_j) \] for \( j=0,1,2,\ldots \).

Proof. (7.4) implies that all the discs \( D_j \) are equivalent to \( D_0 \) and to each other. Therefore \( |M_{j+1} - M_j| ≤ r_j + r_{j+1} \) for all \( j \). On the other hand,
\[ (i.e.(D_j)) r_{j+1} = r_{k,M}(M_{j+1}) \]
\[ r_{k,M}(M_j) - |M_{j+1} - M_j| = (i.e.(D_j)) r_j - |M_{j+1} - M_j| \] for all \( j \). It follows that
\[ (i.e.(D_{j+1}) + 1)/(i.e.(D_j)) - 1 \geq r_j/r_{j+1} \]
for all \( j \). Apply (7.5) and deduce that
\[ (i.e.(D_{j+1}) + 1)/(i.e.(D_j)) - 1 \geq (1+\nu_j)^2/(128n^2) \] (7.6)
for all \( j \). Recall that \((1+\nu_j) = i.e.(D_j) \) and observe that \( i.e.(D_j) \) grows more rapidly than \((1+\nu_j)^2 \) as \( j \) grows. (The latter fact follows from the comparison of (7.5) with the last equation of (7.4) for \( j=0,1,\ldots \), and from the inequality of (7.2).) Q.E.D.

Next assume that (7.5) does not hold for some \( j \). Note that
\[ i.e.(p(x),D_j) = k, r_{k,M}(M_{j+1}) < r_{j+1} < 64n^2 r_{k,M}(M_{j+1}). \]  
(7.7)

and Corollary 7.1 together imply the following estimate,
\[ d(D_{j+1}) ≥ r_{k,M}(M_{j+1}) / ((1+\nu_j)^2r_{j+1}) \]
\[ \alpha_j ≤ \epsilon_{j+1} / r_{k,M}(M) ≤ \epsilon_{j+1} / (r_{k,M}(M_{j+1}) - \epsilon_{j+1}). \]
Therefore
\[ d(D_{j+1}) \geq (r_{n-4}(M_{j+1}) - \epsilon_{j+1})k/(k-1). \]  
(7.8)

If (7.5) does not hold for some \( j \), then
\[ 128n\epsilon_{j+1} = 128n^2(1+\nu_j)^{-2}t_j \leq r_{j+1} < 64n^2r_{n-4}(M_{j+1}), \]
\[ 2t_{j+1} \leq r_{n-4}(M_{j+1}), \]
\[ r_{n-4}(M_{j+1}) - t_{j+1} > r_{n-4}(M_{j+1})/2. \]

Then (7.7) and (7.8) imply that
\[ d(D_{j+1}) \geq (0.5r_{n-4}(M_{j+1})k/(k-1)) > (r_{j+1}/128 n^2)k/(k-1)). \]
(7.9)

On the other hand, unless (7.5) holds, Algorithm 7.1 requires to apply Algorithm 6.1 to the square \( S(M_{j+1}, r_{j+1}/\sqrt{2}) \), superscribing the disc \( D_{j+1} \); in that case the output will satisfy the relations (8.2) is \( O(\log n) \) recursive applications of Subalgorithm 6.1 (due to (7.9)), and then the computations of Algorithm 7.1 will end. Summarizing we arrive at the result.

**Proposition 7.4.** Let \( b \) satisfy (8.5) for \( R \geq r \). Then for the cost \( O(\log(bn\log n)) \) Algorithm 7.1 contracts its input disc \( D(X,R) \) satisfying (7.8) into a disc of radius \( \epsilon \) or else (only for \( k \geq 1 \)) into a disc \( D_{j+1} \) satisfying (7.8). In the latter case, \( O(\log n) \) further recursive applications of Subalgorithm 6.1 starting with the square \( S(M_{j+1}, r_{j+1}/\sqrt{2}) \) suffice in order to satisfy the relation (8.5) for the additional cost \( O(\log\log n)) \).

**8. Computing the Center of Gravity of a Set of Polynomial Zeros.**

Extending (3.6) we arrive at the following formula, [14],
\[ M = \frac{1}{2\pi} \int \left( x \frac{p'}{p}(x) \right) dx, \]
(8.1)

where \( \omega = \sqrt{-1} \), the value \( M = M(U) \) is defined by (7.1), and the domain \( U \) bordered by the contour \( \Gamma \) contains exactly \( k \) zeros of \( p(x) \) (not necessarily distinct), that is, \( x_{1\delta}, \ldots, x_k \delta \). Assuming that \( U = D(X,R) \) is a disc with isolation ratio \( 1 + \nu \), we will choose \( \Gamma \) being the boundary of the disc \( D(X,R), R = (1+\nu)R \) and will approximate to the integral (8.1) using the following sum
\[ M' = \frac{1}{2\pi i} \sum_{q=0}^{Q-1} \sum_{h=0}^{k-1} x'h^q \left( X + R\omega^q \right) p' \left( X + R\omega^q \right) / p(X + R\omega^q), \]
(8.2)

Here \( \omega \) is a primitive \( Q \)-th root of 1, \( \omega^Q = 1, \omega \neq 1 \) for \( 0 < q < Q \). We bound the error of the approximation to \( M \) by \( M' \) basing on the Laurent expansion
\[ p'(x)/p(x) = \sum_{m=0}^{\infty} s_m x^{m+1} - \sum_{m=1}^{\infty} S_m x^{-m}, \]

\[ s_m = \sum_{j} x_j^{m}; S_m = \sum_{j} x_j^{-m} \]

\( \{x_{1\delta}, \ldots, x_{k\delta}\} \) is the set of all the zeros of \( p(x) \) numbered such that \( |X - X_j| < R \) if and only if \( j \leq k \), compare [27], Sec. 12. (8.1)-(8.3) immediately imply that
\[ \left| M' - M \right| \leq 2R(\frac{Q+1}{2} + (n-k)Q^{-1}) / (k(1-\nu)), \]
(8.4)

\[ g = \min_{1 \leq i \leq 2} \min_{1 \leq j \leq 2} \{ |X - x_j| / R, R / |X - x_j| \}, \]

(8.5)

and \( x_1 \ldots, x_n \) denote all the zeros of \( p(x) \). \( g \leq 1/(1+\nu) \) since \( R = (1+\nu)R, \) i.e., \( D(X,R) \geq (1+\nu)^3 \). (8.4) and (8.5) imply that
\[ \left| M' - M \right| \leq 4Rn / (k(1+\nu)^2). \]

We will keep \( Q \) of an order \( c^n \) for a constant \( c \), so the cost of the integration will be \( O(\log n, n^2) \); we will choose the constant \( c \) such that \( M' - M \leq (1+\nu)^{-\epsilon} \), as this is required in Proposition 7.1. Q.E.D.

**9. Turan-Weyl's Isolation Algorithm for Computing All the Zeros of a Polynomial.**

Using the machinery of the previous sections we will now arrive at the bounds of Table 1.1 for computing all the zeros of \( p(x) \). We will proceed recursively; in each recursive step the set of the zeros of \( p(x) \) will be covered with a square or with several squares of the same size.

Initially cover all the zeros of \( p(x) \) with the single square \( S(0,R) \),
\[ R = 2\sqrt{2} \max_{k=1,\ldots,n} |c_{n-k}/c_k|^{1/3}, \]

so \( i.r(S(0,R)) = \infty \), [14], Sect. 6.4. We will maintain isolation ratio at least six for the output squares (and thus for the input squares) of each recursive step. This will enable us to partition each recursive step into parallel processes associated with each isolated square and performed independently of each other. Let \( S(Y,R) \) denote an input square of some recursive step such that \( i.r(S(Y,R)) \geq 6 \). Applying Algorithm 7.1, we will contract the square \( S(Y,R) \) and will either approximate all the \( k \) zeros of \( p(x) \) in \( S(Y,R) \) with errors \( \leq \epsilon \) or will arrive at the case where Subalgorithm 6.1 has been applied and its outputs satisfy (6.2), in particular, in that case \( S(Y,R) \) shrinks to a square \( S \) such that \( r.r(S) > 0.1 \). Then we will apply (Turan-Weyl's) Algorithm 5.1 stopping in \( J = \lceil \log(nR/\epsilon) \rceil + 3 \) iterations. We will, however, end the computations in \( j \) iterations for \( j < J \) if we can group the suspect squares output by iteration \( j \) into at least two maximal connected components, strongly isolated in the sense to be defined now.

Partition the union of all the suspect squares returned by iteration \( j \) into \( H(j) \) maximal connected components, \( C_1,\ldots,C_h(j) \); each component contains at least one zero of \( p(x) \), so
\[ H(j) \leq k. \]

For every \( g \) apply Algorithm 2.1 to the set of all the vertices of all the suspect squares of \( C_g \) and arrive at a square \( S(X_g, R_g^*) \) covering \( C_g \). Note that
\[ R_g^* \leq (k+2)R / 2l+1. \]

Call component \( C_g \) and square \( S(X_g, R_g^*) \) strongly isolated if
\[ |X_g - X_k| > \sqrt{2} (R_g^* + 6 R_g^*) \]

(9.4) implies that the square \( S = S(X_g, R_g^*) \) is equivalent to \( C_g \) and
\[ i.r(S) \geq 6. \]

Let exactly \( h(g) \) strongly isolated components be output by iteration \( g \) of Algorithm 5.1, let \( h(1) = 1 \) for \( g < j, h(j) > 1 \). Combining (9.2)-(9.4) we deduce that
\[ j \leq \left\lceil -0.5 + \log_2 7 + \log_2 ((k+1)(k+2)) \right\rceil. \]

Having completed that iteration \( j \), fix all the \( h(j) \) strongly isolated components and continue applying Algorithm 5.1 to all the suspect squares of all other components until only strongly isolated squares are returned. We will call that modification of Algorithm 5.1 Turan-Weyl's isolation algorithm, compare [25]. Its cost is \( O(H \log n, n) \) where \( H \) denotes the total number of all
the suspect squares processed in all the iterations.

Next we will prove that \( H = O(h) \) where \( h \) denotes the number of strongly isolated components output by the final iteration. Moreover we will prove that \( H = O(h) \) even for a modification of the algorithm where the suspect squares of each strongly isolated component are subdivided further as long as the diameter of the component exceeds the diameter of a suspect square more than twice. Then each output component consists of not more than four suspect squares. Certainly the cost of the original algorithm may only increase due to that modification. To show that \( H = O(h) \), we will retrace back the process of the recursive subdivision of suspect squares, beginning from its end, that is, from the last iteration, which returns \( h \) strongly isolated components. We will respectively reverse the basis property of the forward subdivision process, that is, a subdivision of a suspect square decreases its diameter by 50%, but that diameter is doubled when we retrace the process back; therefore every backtrack step expands the components in all directions. (Exception: the strongly isolated output components will stay unchanged by the backtrack steps where they remained unchanged by the associated steps of the forward process.) The distance between every two components output by iteration \( j \) is lower bounded by the length of an edge of a suspect square; we may at least double such a bound unless in a backtrack step (from iteration \( j \) to iteration \( j-1 \)) these two components are output components or meet each other. Therefore each component \( C \) either is a strongly isolated output component or meets another component in at most \( \lceil \log_2(3k(C)) \rceil \) backtrack steps, where \( k(C) \) is the number of suspect squares in that component \( C \), compare (9.4). Let us represent all the components in all iterations by the nodes of a tree whose \( h \) leaves correspond to the \( h \) output components of the algorithm and whose each edge represents one or several backtrack steps needed in order that one component could meet another. The total number of the nodes of the tree is at most \( 2h-1 \), which also means at most \( h \) edges of the tree. At the leaves level there are at most \( 4h \) suspect squares. This immediately implies that \( H = O(h \log^2 h) \) since the number of suspect squares cannot grow in the backtrack process, which in particular bounds the number of suspect squares in each component by \( h \). The stronger bound \( H = O(h) \) follows from the simple observation that in each step of the backtrack process each connected set of \( g \) suspect squares is imbedded into a set of at most \( 2g/2 \) larger suspect squares; therefore the number of suspect squares processed in all the components has at least five suspect squares decreases at least by 10% in each backtrack step. Consequently a total number of suspect squares in all steps is less than \( 40h \), not counting the suspect squares in the components consisting of at most five suspect squares. If a component consists of \( k \leq 5 \) suspect squares, then the edge in the tree from that component in the direction to the root corresponds to at most \( \lceil \log_2(3k) \rceil \leq 4 \) backtrack steps and therefore to at most \( 20 \) suspect squares. There are at most \( 2h-2 \) edges in the tree, so we arrive at the rough upper bound \( H < 40h+20(h-2) < 80h \).

Summarizing, in \( H = O(h) \) iterations for the overall cost \( O_{\text{a}}(h \log n, a) \), Turan-Weyl's isolation algorithm returns \( h \) strongly isolated components \( C_g, g = 1, \ldots, h \), each consisting of at most four suspect squares. We cover each of these \( h \) components \( C_g \) by a square \( S_g \) equivalent to \( C_g \) and such that \( \text{r}(S_g) \geq 6 \), see (9.5), and compute the indices of \( p(x) \) in all those squares (see Proposition 4.1). Then again we recursively apply Subalgorithm 6.1, Algorithm 7.1, and finally Turan-Weyl's isolation algorithm to each of those squares \( S_g \), until we compute all the zeros of \( p(x) \) with absolute errors less than \( \epsilon \). To estimate the overall cost, associate the subdivision of the input components for each application of Turan-Weyl's isolation algorithm with the edges of the tree, whose nodes are those input components, whose root is the input square \( S(0,R) \) for \( R \) satisfying (9.1), and whose leaves are the components of diameters \( < \epsilon \). There are at most \( n \) leaves, so there are at most \( 2n-1 \) nodes in the tree, and all the required applications of Turan-Weyl's isolation algorithm have overall cost \( O_{\text{a}}(n \log n, a) \). Due to the recursive applications of Algorithms 4.1 and 7.1 (required \( O(n) \) times) and of Subalgorithm 6.1 (required \( O(n \log n) \) times), the overall cost bound increases to \( O_{\text{a}}(n \log n^2 + 2n b, a) \).

**Theorem 9.1.** All the zeros of a polynomial \( p(x) \) of (2.1) can be computed with absolute errors at most \( \epsilon \) for the cost \( O_{\text{a}}(n \log n^2 + 2n b, a) \) where \( b \) satisfies (2.2) for a positive \( R \) that upper bounds the moduli of all the zeros of \( p(x) \).

10. Computing a Single Zero of a Polynomial

Next we will compute a single zero of \( p(x) \) for the cost estimated in line 2 of Table 1.1.

The first two strongly isolated components \( C_1 \) and \( C_2 \) are computed in at most \( O(\log n) \) iterations of Turan-Weyl's algorithm, see (9.1)-(9.6). Compute the indices \( i_1 \) and \( i_2 \) of \( p(x) \) in both of these components, see Proposition 4.1. \( i_1 + i_2 \leq n \), so if \( |i_1 - i_2| \leq n/2 \), say \( i_1 = \lfloor p(x), C_1 \rfloor \leq n/2 \). Apply our algorithms of Sections 6-9 to the component \( C_1 \) and repeat that process recursively, defining a sequence of strongly isolated components \( C_g(0) \geq C_g(1) \geq \ldots \), such that \( C_g(0) = C_1 \),

\[
i(p(x), C_g(0)) \leq a/2^b, \quad b = 0, 1, \ldots,
\]

(10.1) implies that the component \( C_g(0) \) contains at most \( n/2^{b-2} \) suspect squares, so the cost of the corresponding Turan test is \( O_{\text{a}}(\log n, n/2^b) \). Slow down that computation to save processors and arrive at the cost bound \( O_{\text{a}}(\log n)^2/2^b, n/2^b / \log n \). Summing in \( b \) from 0 to \( \log n \), add the cost of \( O(\log a) \) applications of algorithms of Sections 4.7, and 8, and arrive at the bounds of Table 1.1.

In the important case where all the zeros of \( p(x) \) are real, the algorithm for computing a single zero of \( p(x) \) can be substantially improved to reach the cost bound \( O_{\text{a}}(\log n, (\log b + \log^2 n), a) \), see [27].


In this section we will reexamine our algorithms assuming a finite precision of computations. We have reduced the evaluation of polynomial zeros to computing root radii via root powering (Algorithm 3.1) and via numerical integration, applied in (8.2) and in the winding number algorithms. All these computations are immediately reduced to polynomial multiplications (to which also the shifts of variable \( x \) can be reduced), polynomial divisions (to which also the solution of triangular Toeplitz systems can be reduced), and to discrete Fourier transforms. Using the known estimates for the errors of performing those basis operations with finite precision and taking into account that Turan-Weyl's algorithm is self-correcting (that is, the errors of each its iteration are corrected in the next iteration), we may estimate the output errors of finite precision computations by our algorithm and its Boolean (bit-operation) complexity. The required basis estimates in the case of polynomial multiplication and division and DFT can be taken from [3, 13], and [27]. The analysis of the errors of finite precision computations is simple or
readily available for numerical integration, see [14], pp. 239-241,
[27], so we will focus on Turán's tests, specifically, on the solution
of Toeplitz systems and on the shifts of the variable. Next we will show (relying on the known error estimates for those
operations, [3, 13]) that the magnification of errors by Turán's
tests may only require to increase the precision of computations
about O(n) times compared with the prescribed output precision,
while such an increase is inevitable in any algorithm for computing
zeros of an arbitrary polynomial, see [16], pp. 74-77, [18]. (It is even easier to show that the precision of computations
required in our case for numerical integration needs not be
higher than that.)

Let us assume that the input coefficients $c_0, c_1, \ldots, c_n$ of $p(x)$ are integers and that
\[ \max_i |c_i| < 2^m, \epsilon = 2^{-m}, b = m + q, \]
(11.1) compare (2.2) and Theorem 1 of [9]. (We may arrive at that case via truncation of the mantissa of $c_0, \ldots, c_n$ and scaling $p(x)$.) Then it can be shown that the computations with the precision $O(bn)$
binary bits will suffice. Let us verify this for both stages of
Turán's test:

a) shift of the variable $x$ by $X$,
b) computing $\max_i |g_i| / n^{1/\log(n)}$, $i=1, \ldots, n$.

Stage a). We will choose the shift values $X$ such that both real and imaginary parts of $2^{\epsilon X}X$ are integers. (This way we still may assure the absolute output error bound $< \epsilon = 2^{-m}$, note also that $|X| < 2^m$.) Then the coefficients of the polynomial $q(y)$ of (3.2) are integer multiples of $2^{\epsilon X}$ (that is, they take the form $h/2^{\epsilon X}$ where $h$ is an integer), so it is sufficient to compute them with absolute errors less than $1/2^{\epsilon X+1}$ and to recover their exact values via rounding-off. We reduce computing the shift to convolution of two vectors whose entries have absolute values $< 2^{\epsilon n}$ (compare (11.1) and [22]), so that $O(n)$ bit-precision of computations will suffice. (Convolution can be reduced to FFT whose error analysis is available, see [13] or [3], p. 194, or alternatively to integer multiplication, [11], whose error estimates are available in [27].)

Stage b). Consider the evaluation of the power sums via the
iterations (3.3) and via solving the system (3.5). Surely the few
required iterations (3.3) (reduced to convolutions and DFTs) cannot
blow-up the errors too much (compare the error estimates from [13] or [27]), so we will only analyze the errors of the solution
of the triangular Toeplitz systems (3.5). For an entry $s$ of the inverse of a $(n\times n)$ unit triangular Toeplitz matrix $T$, we have the following useful estimate from [3], Lemma 2, p. 192,
\[ |s| \leq O(n \log(t+1)), \]
\[ \log(|s|^{1/\log}) = O(\log(t+1)), \]
where $t$ denotes the maximum absolute value of an entry of $T$. In fact the diagonal entries of the coefficient matrix of the system (3.5) equal $c\pm \epsilon$, which is the $N$th power of the leading coefficient of the polynomial $q(y) = y^p + 1 + y, \epsilon = 3.2$.) We will keep our previous assumptions (see part a) above) that the coefficients of $p(x)$ are $m$-bit integers and that $X$ has real and imaginary parts of the form $1/2^{\epsilon X}$ for an integer $h$, so all the coefficients of $q(y)$ are integer multiples of $1/2^{\epsilon X}$, and it suffices to scale the system by $2^{\epsilon X}$, which means the increase of the precision by $2^{\epsilon X}n = O(bn)$ binary bits. Thus, due to the estimate from [3], it suffices to compute with the precision of $O(bn)$ bits in order to control the errors in Stage b. (We could arrive at a similar result if we used Cauchy's integrals [3.6] in order to compute $\int g_i \, dx_i$, compare [27], p. 54.) Summarizing we obtain that the computations with $O(bn)$ binary bit precision suffice to support the applications of Algorithm 3.1 (and in fact similarly to support all other steps of our algorithms).

The errors of finite precision computation of polynomial zeros are closely related to the errors due to the perturbations of the coefficients. The known bounds on the perturbation errors, see [16], pp. 74-77; [2]. Theorem 4, [18, 27], suggest that we estimate $O(bn)$ for the bit-precision cannot decrease if applied to all the polynomials $p(x)$ of degree $n$. For many input polynomials, however, the perturbation bounds of order $bn$ are overly pessimistic, exceeding the actual values by a factor of $n$, and that property is translated to our precision bounds. Similarly the precision of computations could be decreased by a factor of $n$ if we only need to factor $p(x)$ numerically, that is, to compute $u_i$ and $v_j$ such that all the coefficients of the polynomial
\[ p(x) = \prod_{i=1}^k (u_i x + v_i) \]
have absolute values $\leq \epsilon$, see [27].

Finally we combine our arithmetic complexity estimates with the known bounds on the Boolean circuit complexity of arithmetic operations over integers modulo $2^k$ (that is, with the estimates $O_d(t,h)$ for the Boolean sequential time and $O_d(t,h)$ for the Boolean parallel cost, where $t(h) = O(h \log h \log \log h)$, see [1, 5, 17, 26, 27]). We apply those bounds with $h = O(bn)$, multiply the entries of Table 1.1 by $t(h)$ or by $\log h$, respectively, and arrive at the Boolean circuit complexity estimates for the problems of computing polynomial zeros. The resulting Boolean complexity bounds coincide within (polynomialistic factors with [yet unproven] estimates stated in [27]). Incorporating the estimates of [27] for the Boolean complexity of polynomial multiplication, DFT, and shifts of the variable and of [3] for polynomial division would probably improve our estimates for the Boolean complexity of computing all the zeros of $p(x)$ by (polynomialistic) factors in $n$.

12. Alternatives, Improvements, and Open Problems

We may modify our algorithm in a number of ways, keeping
the same overall estimates for the asymptotic arithmetic complexity (within logarithmic factors). Let us briefly list some of them.

1) Instead of the center of gravity $M$ of $k$ zeros of $p(x)$ in a
disc $D$, computed via (8.1), (8.2), we could compute the zero
of the $(k-1)$-th derivative of $p(x)$ in $D$ via Newton's iteration,
provided that $r(D) > 80n^2$; $z$ may serve similarly to $M$; the cost of computing $z$ is $O_d(\log n, n \log n)$, which
is only slightly worse than the bound of Proposition 7.1, compare
[24, 25].

2) Turán's test can be replaced by the root radius algorithm
of [27], Sect. 15, whose arithmetic cost, is $O_d(n \log n \log n)$,
slightly higher than Turán's; the proof of that cost bound in [27]
is elementary. In the applications of Section 6, we can see some other alternatives to (but not improvement of) Turán's Algorithm 3.1.

3) Using the algorithms of [27], we may effectively compute the
coefficients of the factors $\prod_{k=1}^n (x - x_k)$ of $p(x)$, given a square
$S = S(Y,R)$ containing exactly those $k$ zeros $x_{i\ell}$, $i=1, \ldots, k$, provided that, say $r(S) \geq 3$. Such an isolation of those factors is obtained via numerical integration over the boundary of the disc $D(Y,2R)$ circumscribing $S$ and via subsequent Newton's iteration. The asymptotic cost seems to remain the same as in our algorithm. The latter approach can be extended to replace Turán-Weyl's algorithm: we may apply the integration already where all the zeros of $p(x)$ are included into two or several discs having
isolation ratio say $\geq 1+1/n$; such discs exist and can be computed for a low cost where Subalgorithm $0.1$ has been applied and has output discs $D_0$ and $D_2$ satisfying (6.2), see [24]. In that approach the error analysis becomes much harder, but apparently the Boolean circuit complexity bound (but not arithmetic bounds!) for computing a single complex zero of $p(x)$ can be decreased by a factor of $n$ or so, compare [27].

The major open problem is computing all the complex zeros of $p(x)$ for the cost $O((\log n)^{O(1)}, o(n))$. It is also interesting (at least theoretically) if the root radius estimate of [29] can be extended to cover also the approximation to the $k$-th root radius of $p(x)$ for $s \geq 1$.

Part 2. Toeplitz Computation and Applications

2. Some Basic Definitions

[T] denotes a set of all $n \times n$ Toeplitz matrices, containing the two subsets $[L]$ and $[U]$ of lower and upper triangular Toeplitz matrices, respectively; $T$, $L$, and $U$ (with or without subscripts, superscripts, and so on) will be our notation for the members of the subsets $[T]$, $[L]$ and $[U]$, respectively. We will write $L = L(x)$, $U = U(y)$ if $x$ is the first column of $L$, and if $y$ is the first row of $U$. Here and hereafter the superscript $T$ indicates transposition, while the superscript $*$ indicates the conjugate transpose. Let $A = a_{ij}$ denote the $(i,j)$-th entry of the $n \times n$ matrix $A$.

3. Auxiliary Results

Proposition 3.1. Given an LU-generator of length $d$ for a matrix $V$, the cost of computing any fixed column of $V$ is $O((\log d)n, d)$. Proofs are straightforward. Let two matrices $V_1$ and $V_2$ be given, then $V_1 + V_2 = (a_{ij})$ and $V_1 - V_2 = (a_{ij})$ for $i \leq j$. A class of $n \times n$ matrices $V = V(x, y)$ is said to have the LU-displacement representation, if $V = \sum_{i=1}^n L_i(x) U_i(y)$, where $L_i \in [L], U_i \in [U]$.

4. Inversion of Very Strongly Diagonally Dominant Toeplitz Matrices

Definition 4.1. The pair of the vectors $x, y$ of (3.1) if $x_i \neq 0$ and the pair of the vectors $x', y'$ of (3.2) if $x_i' \neq 0$ will be called the canonical generator of $T^T$.

Corollary 4.1. Under the assumptions of Theorem 4.1, all the diagonal entries of the matrix $T^{-1}$ can be computed from its canonical generator for the cost of $O_A(\log n, n)$.

For $s_2 = 1$, let $A = a_{ij}$. A is called strongly diagonally dominant if for a constant $q$,

$$||A|| < q < 1.$$  (4.1)

Proposition 4.1, see, e.g., [19]. Let (4.1) hold and let Newton's iteration for computing $A^{-1}$ be defined as follows,

$$B_0 = I, B_i = B_{i-1} - B_{i-1} A B_{i-1}, i = 1, 2, $$  (4.2)

Then

$$||B_i - A^{-1}|| \leq ||B_0 - A^{-1}|| < q^{2^i},$$

$$||B_i - A^{-1}|| < (1+q)q^{2^i}$$  (4.3)

for all $i$.

Next assume that $A = [T]$. Let $A = T \in [T]$ and modify Newton's iteration (4.2) as follows,

$$B_1 = 2I - T, B_i = B_{i-1} - B_{i-1} T B_{i-1}, i = 2, 3, ...$$  (4.3)

Here $x$ and $y$ denote the first and the last columns of the matrix $B_i$. The matrix $y$ has the first and the last columns equal to $x^T$ and $y^T$ respectively.

Theorem 4.2. Let (4.1) and (4.2) hold for $T = A$ and $q = 1/10$; denote $||B_i - T^{-1}|| = a(i).$ Then

$$a(i+1) < a(i) \leq 1/1000.$$  (4.4)

then Proposition 4.2 implies that $a(i+1) < a(10)/9 < 2(9a(i)+20)(1-9a(i))$. Let $q = 1/10$. Note that $x$ and $y$ denote the same matrix as $A = \lambda$. Therefore $a(2) < 1/9000$, see Proposition 4.1. If

$$a(i) < 1/9000,$$  (4.5)

then Proposition 4.2 implies that $a(i+1) < a(i)/9 < 2(9a(i)+20)(1-9a(i))$.

For $i = 2, 4, 6$ and therefore (4.5) hold, which implies (4.4) and (4.5) for $i = 3$ and then recursively for $i = 4, 5, 6, ...$, so that

$$34a(1) < 1/250(1/250)$$  (4.6)

Consequently $B_i$ and therefore also $B_i$ converge to $T^{-1}$ quadratically as $i \rightarrow \infty$.

On the other hand, $T$ and $B_i$ for all $i$ are in $[L]_d$, so the cost of each iteration of (4.3) is $O_A(\log n, n)$, see Propositions 3.1 and 3.2, and we arrive at the following result.

Theorem 4.1. Let $d$ be a constant, $A = T$ be an $n \times n$ Toeplitz matrix satisfying (4.1) for $q = 1/10$. Then the canonical generator $x, y$ of (3.1) of the inverse matrix $T^{-1}$ can be computed with error norm $\leq 1/250^2$ for the cost of $O_A(\log n, n)$. 

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5. Computing Matrix Powers

Definition 5.1. The canonical set of entries of a square matrix is the set of all its entries in its first and its last columns and on its diagonal.

Next let us compute the canonical sets of entries of the powers \( T^k \) of a given \( n \times n \) Toeplitz matrix \( T \) for \( k = 0, 1, \ldots, q \) where \( q \geq n \); in this paper we will only use \( q = n \) and (in Appendix) \( q = 2n \). Let \( m = 2q \), \( \omega \) be a primitive \((m+1)\)th root of 1, \( h \) be such that

\[
0 < h \|T\| < 1/10, \quad h_j = h^{2j} / \omega^{j}, \quad j = 0, 1, \ldots, m. \tag{5.1}
\]

Then (4.1) holds for \( A = I - h_j T \) for each \( j \), so we may apply the algorithm (4.3), Theorem 4.1, and Corollary 3.2 and compute the canonical set of entries of the matrix

\[
(1 - h_j T)^{-1} = \sum_{k=0}^{\infty} (h_j T)^k = \sum_{k=0}^{\infty} (h T)^k \omega^k \tag{5.2}
\]

with absolute error bound

\[
E^* = (1/34)/250^c \tag{5.3}
\]

where \( c \) is an arbitrary constant for the cost \( O_A(\log^2 n, n) \).

Let \( m = 2q \), \( S_j = \sum_{k=0}^{\infty} (h T)^k \omega^k \). Then

\[
E = \| ((1 - h_j T)^{-1} - S_j) \| \leq (h \|T\|)^m < 1/10^m. \tag{5.4}
\]

see (5.2). The entries of \((1 - h_j T)^{-1}\) approximate to the entries of \( S_j \) with absolute errors \( E^* + E \), see (5.3), (5.4). The cost of computing all those approximations for all \( j \) is \( O_A(\log^2 n, m) \).

Next apply inverse FFTs in order to compute the canonical set of entries of \((h T)^k\) for all \( k \leq m \) (FFTs are applied once for each entry of the canonical set), and then divide the results by \( h^k \) to obtain the desired entries of \( T^k \). The divisions by \( h^k \) increase the absolute errors \( h^k \) times, and \( k \leq q = m/2 \) in our case.

Theorem 5.1. The canonical set of entries of the powers \( T^k \) of an \( n \times n \) Toeplitz matrix for \( k = 0, 1, \ldots, q \geq n \), can be computed with absolute errors less than

\[
E = O(1/250^c h) (n \log^2 n) / 10^m \tag{5.5}
\]

for the cost \( O_A(\log \log n, n^2) \) using the precision of computations of \( O_A(\log(n^2), n) \) binary bits, where a positive \( h \) satisfies (5.1), \( | \log h | = O(\log|T|) \), and \( c \) is an arbitrary constant. If all the entries of \( T \) are integers and if the error bound \( E \) of (5.5) is less than \( 1/2 \), then the exact values of the entries of the canonical sets of \( T^k \) can be recovered for all \( k \) via rounding-off the computed approximations to the nearest integers.

6. Canny-LeVerrier Algorithm for Toeplitz Computations

Recall the following definitions and facts.

\[
s_k = \text{trace}(T^k) = k = 1, \ldots, n, \tag{6.1}
\]

\[
\bar{c} = (D + I)^{-1} \bar{c}, \tag{6.2}
\]

\[
D = \text{diag}(1, 2, \ldots, n), \quad L = \text{diag}(L), \quad \bar{c} = [c_0, \ldots, c_{n-1}]^T ;
\]

\[
\det(\lambda I - T) = \lambda^n - \sum_{j=1}^{n-1} c_j \lambda^j = \prod_{j=1}^{n} (\lambda - \lambda_j), \quad c_0 = (-1)^{n-1} \det(T); \tag{6.3}
\]

\[
T^{-1} = (T^{n-1} - \sum_{j=1}^{n-1} c_j T^{j-1})/c_0. \tag{6.4}
\]

Algorithm 6.1 (Caney-LeVerrier), \[10\]. Input:

\( T^k, k = 1, 2, \ldots, n \).

Successively compute the vectors \( \bar{c} = [c_0, c_1, \ldots, c_{n-1}]^T \) of (6.1) and \( \bar{c} = [c_{n-2}, c_{n-1}, \ldots, c_0]^T \) of (6.2) and then the matrix

\[
\text{adj}(T) = \pm (T^{n-1} - \sum_{j=1}^{n-1} c_j T^{j-1}). \tag{6.6}
\]

Then define \( T^{-1} \) as the pair \((\text{adj}(T), \det(T))\) or if \( c_0 \neq 0 \), compute

\[
T^{-1} = \text{adj}(T)/\det(T). \tag{6.7}
\]

Applying Algorithm 6.1 in the case of a Toeplitz matrix \( T \), we take the canonical sets of entries of \( T^k \) as the inputs and only compute the first and the last columns of \( \text{adj}(T) \) and of \( T^{-1} \) or, if \( x_0 = 0 \) in (3.1), we imbed \( T \) into the \((n+1) \times (n+1) \) matrix \( T' \) of Theorem 3.1 (choosing the entries \( t_{00}^T \) and \( t_{0n}^T \) of \( T' \) such that \( T' \) is nonsingular) and repeat the computation with \( T' \) replaced by \( T \). Computing the vector \( \bar{c} \) that satisfies (6.2) can be reduced to the inversion of the triangular matrix \( D + L \) for the cost of \( O_A(\log^2 n, M(n)) \), but we use the elegant algorithm of \[27\], sect. 13, in that stage and arrive at the following cost bounds.

Table 6.1.

<table>
<thead>
<tr>
<th>( O_A(\log^2 n, n^2) )</th>
<th>( O_A(\log^3 n, n / \log n) )</th>
<th>( O_A(\log n, n^2/ \log n) )</th>
</tr>
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</table>

7. Extensions to Computations with Different Structured Matrices and to Computing the Greatest Common Divisor of Polynomials and the Rational Interpolation

Our algorithms and complexity bounds can be immediately extended to computations with many well structured matrices, such as all the matrices of the class (2.1) (in that case the asymptotic parallel time bounds do not change and the processor bounds increase d times), the matrices of similar classes, defined by Hankel type displacement generators or by some other similar operators, and so on, see \[9, 12\]. If a matrix \( V \) belongs to such a class of structured matrices, then so are the matrices \( V^T \) and \( V^TV \), see \[9\] (which implies immediate extensions of our results to least-squares computations with structured matrices); furthermore our algorithm can be easily extended also to the case of structured block matrices. In fact, our approach works whenever Theorem 3.1 and Proposition 3.2 can be extended, and numerous extensions of those theorem and proposition are presented or implicit in \[9, 12\]. In particular this implies

Theorem 7, \[23\]. The greatest common divisor of two given polynomials of degrees at most \( n \) with integer coefficients whose absolute values are less than a fixed value \( t \) can be computed for the cost of \( O_A(\log^2 n, n^2) \) using the precision of computations of \( O(\log nt) \) binary bits.

Similarly our approach can be extended to several other rational and polynomial computations (such as the evaluation of the gcd of several polynomials, of the least common multiple of two or several polynomials, of the entries of the extended Euclidean scheme, of the squarefree decomposition of polynomials, of elementary symmetric functions, of partial fraction decomposition, of rational interpolation (Hermite or Cauchy) and of Padé approximation) basing on the known reduction of those problems to solving Toeplitz linear systems, see \[7, 30, 31\].
Appendix. Fast Sequential Inversion of Toeplitz and Almost Toeplitz Matrices

We may modify the method of Section 4 and extend it to $p$-adic computations for the inversion of integer matrices of the class $[LU]_d$, which will lead us to their effective sequential inversion (although its parallel Boolean time is at best linear). The algorithm can be extended to the inversion of other structured matrices.

Proposition A.1 Let $S$ be a nonsingular $n \times n$ integer matrix from the class $[LU]_d$ for a natural constant $d$, $k$ be a positive integer, $p > a$ be a prime in the interval

$$n^d||T||^{1/k} < p < n^{d+1}||T||^{1/k}$$

for a positive constant $c$,

$$p^N \geq (a||T||)^{1/k} > p^{N/2} \text{ for } N = 2^k.$$

Then the matrix $S \text{ mod } p$ is nonsingular with probability converging to 1 as $n \to \infty$, and the matrix $S^{-1}$ can be computed for the overall sequential Boolean cost $O_p(n \log(n))\mu(b))$.

References


