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by

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Strong k-connectivity in digraphs and random digraphs.

1 SUMMARY

This paper concerns an extension of the strong connectivity notion in directed graphs. A digraph D is k-strongly connected if, for each x,y vertices of D, there exist $\geq$ k vertex disjoint paths from x to y and also $\geq$ k vertex disjoint paths from y to x. A k-strong block of a digraph D is a maximal k-strongly connected subgraph of D. We show here how many results about the k-blocks in undirected graphs extend to k-strong blocks in digraphs. (Separation lemma, overlapping of k-strong blocks, number of them, see [MATULA, 78].) We prove, for example, that the maximum number of k-strong blocks for all $k \geq 1$ in any n-vertex graph is $\lfloor (2n-1)/3 \rfloor$. We also prove that two k-strong blocks cannot have more than k-1 vertices in common. We furthermore present results bounding the cardinality of the biggest k-strong block in random digraphs of the $D_{n,p}$ model. We show here that the cardinality of the biggest k-strong block is $\geq n - \log n$ with probability $\geq 1 - n^{-(c_1(k)\frac{1}{2}-k)}$ for $p \geq \frac{c_1(k)}{n}$ and $c_1(k) \geq 2k + 4$. We also show that if $p \geq c(k) \frac{\log n}{n}$ with $c(k) \geq 16k^3$ then the digraph $D_{n,p}$ is k-strongly connected with very high probability ($\geq 1 - \frac{1}{n^{d'(k)}}$ with $d'(k) > 1$). This work generalizes previous work of [REIF, SPIRAKIS, 81] on random undirected graphs.

2 INTRODUCTION

A *digraph* $D = (V,E)$ consists of a finite nonempty set V of *vertices* together with a prescribed subset E of $V \times V - \{(u,u): u \in V\}$ (set of directed *edges*). (We allow no loops neither multiple edges.) A digraph D is *k-strongly connected* if, for each x,y vertices of D, there exist $\geq$ k vertex disjoint paths from x to y and also $\geq$ k vertex disjoint paths from y to x. D has *strong connectivity k(D)* = k if D is k-strongly connected but not K + 1 - strongly connected. A *k-strong block* of a digraph D is a maximal k-strongly connected subgraph of D. A k-strong block is *trivial* if it has only one vertex. We extend here the definitions of [MATULA, 78] for k-connectivity and k-blocks in digraphs in a natural way. k-strong connectivity seems to be an interesting property of a graph, in addition to being a natural extension of a mathematical structure. In [KLEINROCK, 72] it is related to message flow in computer networks. The so-called association graphs used in sociology and data cluster analysis may use the theory of strong k-connectivity ([MATULA, 77], [JARDINE, SIBSON, 71]). We give here alternative characterization theorems of the k-strong blocks. We prove various structural properties of k-strong blocks, namely limited overlap, the k-strong block separation lemma (providing also an $O(n^4)$ algorithm for finding all k-strong blocks in an n-vertex digraph) and we provide an achievable upper bound on the number of k-strong blocks for all $k \geq 1$ in any n-vertex graph. (This bound is equal to $\lfloor(2n-1)/3\rfloor$.) All these results are generalizations and extensions of the corresponding results of [MATULA, 78] on k-blocks in undirected graphs.

We also examine k-strong connectivity in the model $D_{n,p}$ of random digraphs, defined precisely as follows: For $0 \leq p \leq 1$ and $n \geq 0$ let $D_{n,p}$ be a random variable whose values are digraphs on the vertex set $\{1,2,\ldots,n\}$. If e = (u,v) and u,v are vertices, then Prob{e is an edge} = p and these probabilities are independent for different ordered pairs e. Extending the previous undirected graph results of [ERDOS, RENYI, 60] and [KARP, TARJAN, 80] for $k = 1,2$ and [REIF, SPIRAKIS, 81] for general k in undirected graphs, we prove that for each constant $k > 0$ and any ε $(0 < \varepsilon < 1)$ and $\alpha > 1$ there is a $c(k,\alpha,\varepsilon) > 0$ such that the random digraph $D_{n,p}$ with $p \geq \frac{c}{n}$ has a k-strong block of cardinality $\geq \varepsilon \cdot n$ with probability at least $1 - e^{-\alpha n}$. We also show that for any $g(n) = o(n)$ there are constants $c(k) \geq 4k$ and $d(k) \geq 2$ such that the size of the biggest k-strong block is $\geq n - g(n)$ with probability $\geq 1 - (\log n)/n^{d(k)}$ for $p \geq c(k)\ (\log n)/n$ An immediate corollary of that is that $D_{n,p}$ is almost surely k-strongly connected for such high values of p. Finally, we prove that for any $g(n) = o(n)$ there is a constant $c_1(k) = \max(3, c(k))$ and a function $t(n) > (c_1(k)\ (\log n))/g(n)$ such that if $p \geq t(n)/n$ then the size of the biggest k-strong block is $\geq n - g(n)$ with probability $\geq 1 - \frac{n}{e^{t(n)g(n)}}$ $\to 1$ as $n \to \infty$. An immediate corollary of that is that $D_{n,p}$ with $p \geq (c_1(k))/n$ has an n-log n size k-strong block with probability $\geq 1 - n^{-c_1(k)+1}$. Similar results were proved for undirected graphs in [REIF, SPIRAKIS, 81].

3 PROPERTIES OF k-STRONG BLOCKS

<u>Proposition 1</u> If D is a digraph and G is the undirected version of D, then $k(D) \leq k(G) \leq 2 \cdot k(D)$, where $k(G)$ is the connectivity of G.

<u>Proof</u> By Menger's theorem an undirected graph is k-connected if every pair of points is joined by at least k vertex-disjoint paths.

<u>Proposition 2</u> Each k-strong block has at least k vertices or it is trivial.

Proof Easy by proposition 1 and the corresponding property of undirected k-blocks (see [MATULA, 78]).

Lemma 1 The minimum number of vertices separating vertex s from vertex t in the direction s to t, is the maximum number of vertex disjoint s to t paths.

For the proof, see the Appendix. It is a modification of Dirac's proof to Menger's theorem.

Theorem 1 The digraph D is k-strongly connected if for every vertex x and for every vertex y, there are vertex cuts from x to y and from y to x of size at least k.

Proof By Lemma 1 and the definition of k-strong connectivity.

Theorem 2 Let D be a k-strongly connected digraph and let x be a single vertex graph with no edges. Let $v_1, \ldots, v_k$ be k distinct vertices of D. Construct the digraph D' which has vertex set consisting of vertices (D) u {x} and edge set the union of the edge set of D and $\{(v_i, x), (x, v_i)$ $i = 1, \ldots, k\}$. Then D' is k-strongly connected.

Proof Immediate by Theorem 1 (see figure 1)).

Theorem 3 Two k-strong blocks B_1, B_2 cannot have more than k-1 vertices in common.

Proof Assume, by contradiction, that they have $\geq$ k vertices in common, $v_1, \ldots, v_h$, $h \geq k$ (see figure 2).

Let x be any vertex of B_1 and y be any vertex of B_2, while neither x nor y is a common vertex v_i, $1 \leq i \leq h$. Then we claim that we cannot find a vertex cut from x to y or from y to x of size $< k$.

<u>Proof of claim</u>: If we could, let S be the set of vertices in the cut, $|S| < k$. Let S_1, S_2, S_c be the intersections of S with $V(B_1) - \{u_1, \ldots, u_h\}$, $V(B_2) - \{u_1, \ldots, u_h\}$ and $\{u_1 \ldots, u_h\}$ respectively. Clearly $|S_1| < k$, $|S_2| < k$, $|S_c| < k$. By taking the set S_c out, at least one of the u_i (call it $\bar{u}$) remains in the digraph. x had $\geq k$ disjoint paths to $\bar{u}$ and hence the removal of S_1 u S_c leaves at least one path from x to $\bar{u}$. Similarly, the removal of S_cv S_2 leaves out at least one path from $\bar{u}$ to y. Similarly for the direction y x. Hence the set S is not a cut set, which contradicts to our assumption.

By using the just proved claim we remark that $B_1 \cup B_2$ should be k-strongly connected if $h \geqslant k$. But this contradicts to the maximality of each of them. QED.

<u>Definition</u> Let D be a digraph (V,E) and let $S \subseteq V$ be a vertex set. With $<S>$ we denote the directed subgraph induced by S on D.

4 STRUCTURE AND ENUMERATION OF k-STRONG BLOCKS

<u>Definition</u> A *separating set* S of the digraph D is a vertex set $S \subset V(D)$ such that $D - S$ is not (one)-strongly connected.

The strongly connected components of $D - S$ are denoted by $<A_1>, \ldots, <A_m>$ where $m \geq 2$.

<u>Proposition 3</u> A *minimum* separating set has $|S| = k(D)$.

Proof By theorem 1, at least $k(D)$ vertices are needed to be removed to disconnect two points x, y in at least one of the directions xy, yx.

Lemma 2 (Block separation lemma) Let $S \subseteq V(G)$ be a minimum separating set of the digraph D (with $\langle A_1\rangle, \ldots, \langle A_m\rangle$, $m \geq 2$the strongly connected components of $D - \langle S\rangle$) and let $k \geq k(D) + 1$. Then each k-strong block of D is a k-strong block of $\langle A_i \cup S\rangle$ for precisely one value of i and each k-strong block of $\langle A_i \cup S\rangle$, $\forall_i$ is a k-strong block of D.

Proof It is immediate for D not strongly connected. Let D be a strongly connected digraph with some minimum separating set S and let $k \geq k(D) + 1$.

Let B be a k-strong block of D. Since $V(B) \cap S$ is not a separating set of B and since $|V(B)| > |S|$, B must be a k-strong subgraph of precisely one strongly connected component, $\langle A_i \cup S\rangle$, of $D - S$, B then is a subgraph of precisely one k-strong block, B^*, of $\langle A_i \cup S\rangle$, and B^* is then a k-strong subgraph of D containing B. But B is maximal with respect to k-strong connectivity in D. Hence $B = B^*$, so B is a k-strong block of $\langle A_i \cup S\rangle$.

For any i, let B^* be a k-strong block of $\langle A_i \cup S\rangle$ with $k \geq k(D) + 1$. B^* then is a subgraph of some k-strong block B of D. Since B cannot be separated by $V(B) \cap S$ we conclude that $V(B) \subseteq V(\langle A_i \cup S\rangle)$. Thus B is a k-strong subgraph of $\langle A_i \cup S\rangle$ containing B^* as a subgraph. By maximality of B^* we get $B = B^*$, proving the lemma. QED

Definition For $n \geq 1$ let $w(D,n)$ be the number of k-strong blocks of D for $k \geq n$. Define $w(D) = w(D,1)$.

It is obvious that, for a strongly connected D

$$w(D) = w(D,k(D)) = 1 + \sum_{i=1}^{m} w(\langle A_i \rangle \cup S,\ k(D) + 1)$$

(decomposition formula)

Lemma 3

$$w(D,n) \leq \lfloor (2(V(D)-n) + 1)/3 \rfloor \text{ for } 1 \leq n \leq V(D) - 1$$

$$= 0 \qquad \text{for} \qquad n \geq V(D)$$

Proof The verification of the above formula is obvious for complete D and for D with $|V(D)| \leq 3$.

By induction, let it hold for all digraphs D with $1 \leq |V(D)| \leq j-1$ and let D_j be a particular noncomplete j-vertex digraph.

Let S be a minimum separating set of D_j with $\langle A_1 \rangle, \ldots, \langle A_m \rangle$, $m \geq 2$, the strongly connected components of $D_j - S$.

Consider three cases depending on n and $w(\langle A_i \cup S \rangle, n)$.

(i) Suppose $n \geq k(D_j) + 1$ and that there is one $i \in \{1, \ldots, m\}$ such that

$$w(\langle A_i \cup S \rangle, n) = 0$$

For $k \geq n$ (from the block separation lemma) the k-strong blocks of D_j are precisely the k-strong blocks of $D_j - \langle A_i \rangle$.

Thus

$$w(D_j, n) = w(D_j - \langle A_i \rangle, n)$$

and since $|V(D_j - \langle A_i \rangle)| \leq j - 1$, the inequality follows by the induction hypothesis.

(ii) For $n \geq k(D) + 1$ for every digraph D, we have from the separation lemma

$$w\,(D,n) = \sum_{i=1}^{m} w\,(\langle A_i \cup S \rangle, n)$$

Let $n \geq |S| + 1 = k\,(D_j) + 1$

Let also $w(\langle A_i \cup S \rangle, n) \geq 1 \quad \forall_i = 1, \ldots, m$

Thus, $|V(\langle A_i \cup S \rangle)| \geq n + 1 \quad \forall_i = 1, \ldots, m$

So

$$w(D_j,n) = \sum_{i=1}^{m} w(\langle A_i \cup S \rangle, n) \leq \sum_{i=1}^{m} \lfloor (2|V(A_i \cup S)| - 2n+1)/3 \rfloor$$

(by the induction hypothesis)

$$\leq \sum_{i=1}^{m} \lfloor (2|V(A_i)| + 2|S| - 2n + 1)/3 \rfloor$$

$$\leq \lfloor (2j + 2(m-1)|S| - 2mn + m)/3 \rfloor$$

$$\leq \lfloor 2j - 2n + 1)/3 \rfloor, \qquad \text{QED}$$

(Note that $2(m-1)|S| - 2mn + m \leq 2 - m - 2n \leq -2n$, since $m \geq 2$.)

(iii) Let $n = |S| = k(D_j) \geq 1$

Then $|\langle A_i \cup S \rangle| \geq n + 1 \quad \forall_i = 1, \ldots, m$

So, by the decomposition formula of page 7 and by the induction hypothesis

$$w(D_j, n) = 1 + \sum_{i=1}^{m} w(<A_i> \cup S, n+1)$$

$$\leq 1 + \lfloor \sum_{i=1}^{m} ((2|A_i| + 2|S| - 2n - 1)/3) \rfloor$$

$$\leq 1 + \lfloor (2j - 2n - m)/3 \rfloor$$

$$\leq \lfloor 2j - 2n + 1)/3 \rfloor \qquad \text{QED}$$

Corollary

$$w(D) \leq \max w(D,1) = \lfloor (2n - 1)/3 \rfloor$$
$$|V(D)| = n$$

We now show that this upper bound is achievable.

Lemma 4 There exists a digraph D_w such that

$$w(D_w) = \lfloor (2n - 1)/3 \rfloor$$

Proof Consider the following digraph D_w.

$$v(D_w) = \left\{ a_1, \ldots, a_{\lfloor \frac{n+2}{3} \rfloor}, b_1, \ldots, b_{\lfloor \frac{n+1}{3} \rfloor}, c_1, \ldots, c_{\lfloor n/3 \rfloor} \right\}$$

Let $E(D_w)$ be the union of the following sets:

$\{(a_i, a_j) \cup (a_j, a_i)\}$ $\quad 1 \leq i \leq j \leq \lfloor (n+2)/3 \rfloor$,

$\{(a_i, b_j), (b_j, a_i)\}$ $\quad 1 \leq i \leq j \leq \lfloor (n+1)/3 \rfloor$,

$\{(a_i, c_j), (c_j, a_i)\}$ $\quad 1 \leq i \leq j \leq \lfloor n/3 \rfloor$,

$\{(b_i, c_i), (c_i, b_i)\}$ $\quad 1 \leq i \leq \lfloor n/3 \rfloor$

For any $1 \leq k \leq \lfloor (n-1)/3 \rfloor$, the subgraph

$D_w - \{b_1, \ldots, b_{k-1}, c_1, \ldots, c_{k-1}\}$ is a k-strong block of D_w and there are no other k-strong blocks for that value of k, except trivial k-strong blocks.

We have also to count the k-strong blocks with no (k + 1)-strong blocks inside. For $n = 0, 1 \pmod 3$ the complete subgraphs $\{b_i, c_i, a_1, a_2, \ldots a_i\}$ $1 \leq i \leq \lfloor n/3 \rfloor$ are the only kind of these k-blocks. For $n = 2 \pmod 3$ we have also to add the clique

$$\{b_{\left\lfloor \frac{n+1}{3} \right\rfloor}, a_1, a_2, \ldots, a_{\left\lfloor \frac{n+2}{3} \right\rfloor}\}$$

So, the maximum number of the k-strong blocks with no (k+1)-strong blocks inside is equal to $\left\lfloor \frac{n+1}{3} \right\rfloor$ for $n \geq 2$.

So the total number of k-blocks in D_w is

$$\left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{2n-1}{3} \right\rfloor \qquad \text{for } n \geq 2$$

5. GIANT k-STRONG BLOCKS IN RANDOM GRAPHS

Theorem 5 For every $\varepsilon \in (0,1)$, $\alpha > 1$ and $k > 0$ there is a $c(k; \alpha, \varepsilon) > 0$ such that, for $p \geq \frac{c}{n}$, the random digraph $D_{n,p}$ with $p \geq \frac{c}{n}$ has a k-strong block of vertex cardinality $\geq \varepsilon \cdot n$ with probability at least $1 - e^{-\alpha n}$.

Proof Let $D = (V,E)$ be an instance of $D_{n,p}$. Let $\mathcal{E}_1$ be the event "D has no k-strong block of cardinality $\geq \tilde{\varepsilon} n$". Assume $\mathcal{E}_1$ be true on D. Construct a digraph H with the k-strong blocks as vertices and an edge from the k-strong block b_1 to k-strong block b_2 only if *there is no* vertex cut of size $\leq$ k-1 separating b_1 from b_2 to the direction $b_1 \rightarrow b_2$. (Note that at least one such vertex cut, either to the direction $b_1 b_2$ or to $b_2 b_1$ exists, and it is of cardinality $\leq$ k-1.) Clearly H is acyclic and hence *not* strongly connected. Let the set S be initially empty. Add to S the k-blocks of D one-by-one, following the reverse topological order of H. Each addition of a k-strong block to S, adds at most (k-1) vertices to the border-set of S (being the set of the vertices of S having edges to the outside of S) and at least one vertex to the rest of S (since each k-strong block has at least k-vertices if it is no trivial) or causes the transformation of a vertex of the border-set of S to a vertex of the rest of S. Thus, at least 1/k of the vertices of S have no edges to the outside of S. Continue the above construction, just until S has cardinality $\geq \varepsilon' \frac{n}{2}$ where $\varepsilon' = \min(\varepsilon, 1-\varepsilon)$. Then (by our assumption that $\mathcal{E}_1$ holds)

$$\varepsilon' \frac{n}{2} \leq |S| \leq \varepsilon' \frac{n}{2} + \varepsilon n$$

So, $$|S - B(S)| \geq \frac{\varepsilon'}{2k} n$$

where B(S) is the border-set of S.

Also,

$$|V(D) - S| \geq n(1 - \varepsilon - \frac{1}{2} \varepsilon') > 0$$

Let $A = S - B(S)$, $B = V - S$.

Then $|A| \geq \varepsilon_1 \cdot n$, $|B| \geq \varepsilon_2 \cdot n$

with $\varepsilon_1 = \frac{\varepsilon'}{2k}$

$$\varepsilon_2 = 1 - \varepsilon - \frac{1}{2}\varepsilon'$$

and no edge exists from A to B.

This event's probability is bounded above by

$$\sum_{\text{all } A,B} \text{Prob}\{\text{no edge from A to B}\}$$

$$\leq \frac{1}{2} \cdot 4^n \cdot (1-p)^{\varepsilon_1 n \varepsilon_2 n}$$

$$\leq \frac{1}{2}(4\, e^{-\varepsilon_1 \varepsilon_2 c})^n \leq e^{-\alpha n}$$

for $p \geq \frac{c}{n}$ and $c \geq \dfrac{\alpha + \log_e 4}{\varepsilon_1 \varepsilon_2}$

and any $\alpha > 1$.

So, $\text{Prob}\,(\mathcal{E}_1) \leq e^{-\alpha n}$ QED.

6 k-STRONG BLOCKS OF DENSE RANDOM DIGRAPHS

This section considers random digraphs of the model $D_{n,p}$ with $p \geq c\,\frac{\log n}{n}$

Theorem 6 For any constant integer $k > 0$ and any n and $m < \frac{n}{2k}$ there are constants $c(k)$, $d(k) > 0$ such that, the cardinality X of the biggest k-strong block of the digraph $D_{n,p}$ with $p \geq c(k)\,\frac{\log n}{n}$ satisfies the proprety

$$\text{Prob}\,\{X = n - m\} \leq n^{-m \cdot d(k)}$$

<u>Proof</u> Let D be an instance of $D_{n,p}$ and let the event $X = n-m$ be true in that instance. Let A be a k-strong block with $|A| = X$. For every $u \varepsilon V-A$, at least one of the following two inequalities holds

$$|(u,v,) \in E(D) : V \in A| \leq k - 1 \qquad (*)$$

$$|(v,u,) \in E(D) : V \in A| \leq k - 1 \qquad (**)$$

So, for at least one-half of the vertices of V-A the same inequality holds (either (*) or (**). This is so, since failure of both (*) and (**) for u would imply that $u \in A$ by theorem 2. Without loss of generality, let (*) be the property holding for $\geq \frac{1}{2}$ of the vertices of V-A. Call the set of these vertices U.

So, $\qquad |U| \geq \frac{1}{2} |V-A| = \frac{1}{2} m$

and $\qquad \forall u \in U \quad |\{(u,v) \in E(D) : V \in A| \leq k-1$

Let $\qquad A_1 = \{v \varepsilon A | \exists u \in U : (u,v) \in E(D)\}$

Then $\qquad |A_1| \leq (k-1) |U| \leq (k-1) \cdot m$

Let $A_2 = A - A_1$. We get $|A_2| \geq n - m - (k-1)m$

or $|A_2| \geq n - km$.

Furthermore, there is no edge from U to A_2.

Let $\mathcal{E}$ be the above event. The probability of $\mathcal{E}$ is bounded above by the

$$u(n,m) = \binom{n}{m} \binom{n-m}{n-km} (1-p)^{(n-km)(m/2)} \qquad (***)$$

(Note the way this upper bound is formed. We use n -km in $\binom{n-m}{n-km}$ since $\binom{n}{x}$ is decreasing for $x > \frac{n}{2}$ and n - km is the minimum value possible $> \frac{n}{2}$.)

We have to use the minimum exponent of (1-p).

But $1-p \leq 1 - c\frac{\log n}{n}$ since $p \geq c\frac{\log n}{n}$

Also $\binom{n-m}{n-km} \leq e^{(k-1)m \log(n-m)}$ since $(k-1)m < \frac{n-m}{2}$

Also $\binom{n}{m} \leq e^{m \log n}$ since $m < \frac{n}{2}$

Finally $1 - c\frac{\log n}{n} \leqslant e^{-c\frac{\log n}{n}} \; \forall n$

So, $u(n,m) \leqslant n^{-d(n,m)}$

where
$$d(n,m) = \frac{c}{2}\left(1 - \frac{km}{n}\right)m - m - (k-1)m\,\frac{\log(n-m)}{\log n}$$
$$\geqslant \frac{c}{2}\,m\left(1 - \frac{km}{n}\right) - m - (k-1)m$$
$$\geqslant \frac{c}{2}\,m - km \qquad \text{(by our assumption)}$$
$$\geq m\,d(k) \qquad \text{where } d(k) = \frac{c}{4} - k$$

Note that $d(k) > 0$ **iff** $c(k) > 4k$

So

$$\text{Prob}(\mathcal{E}) \leq n^{-m\cdot d(k)}$$ QED.

<u>Theorem 7</u> For any constant interger $k > 0$ and any $n >> k$ there is a constant $c(k) > 0$ and a $d(k) > 0$ such that the cardinality X of the biggest k-strong block of the digraph $D_{n,p}$ with $p \geq c(k)\frac{\log n}{n}$ satisfies the property

$$\text{Prob}\ \{X \leq n - \log n\} < 2n^{(1 - \log n\ d(k))}$$

<u>Proof</u> We have (by using theorem 6)

that $$\text{Prob}\left\{\log n \leq n - X < \frac{n}{2k}\right\} = \sum_{m = \log n}^{n-2k} n^{-m \cdot d(k)}$$

with $$d(k) = \frac{c(k)}{4} - k > 0 \text{ for } c(k) > 4k$$

So

$$\text{Prob}\left\{\log n \leq n - X < \frac{n}{2k}\right\} < n \cdot n^{-\log n \cdot d(k)}$$

$$< n^{1 - \log n \cdot d(k)}$$

Also, from theorem 5, and by using

$\varepsilon = \frac{1}{2k}$, we get

$$\text{Prob}\ \left\{n - X > \frac{n}{2k}\right\} < e^{-\alpha n}$$

for any $\alpha > 1$ and $c(k) \geq \frac{\alpha + \log_c 4}{\varepsilon_1 \varepsilon_2}$

and $$\varepsilon_1 \varepsilon_2 = \frac{1}{4k^2} \cdot \left(1 - \frac{3}{4k}\right)$$

So, for

$$c(k) > \max\left(4k, \frac{\alpha + \log e4}{\varepsilon_1 \varepsilon_2}\right)$$

$$(\text{or } c(k) > 16k^3)$$

we get

$$\text{Prob}\ \{\log n \leq n - X\} < e^{-\alpha n} + n^{1 - \log n \cdot d(k)}$$

or

$$\text{Prob}\ \{X \leq n - \log n\} < 2\ n^{1 - \log n \cdot d(k)}$$

for sufficiently large n.

QED.

NOTE

Theorem 7 says that, for $p \geq c(k)\ \frac{\log n}{n}$ the digraph $D_{n,p}$ has a k-strong block with prob→1 as n→∞.

Theorem 8 For any constant integer k > o and n >> k there are constants $c(k) > 0$, $d'(k) > 1$ such that the random digraph $D_{n,p}$ with $p \geq c(k)\ \frac{\log n}{n}$ is k-strongly connected with probability

$$\geq 1 - 2n^{-d'(k)}$$

Proof Let R = n - X, X = cardinality of the biggest k-strong block of $D_{n,p}$. By using theorems 5,6 and $c(k) > 2 + \max\left(4k, \frac{\alpha + \log 4}{\varepsilon_1 \varepsilon_2}\right)$ with $\varepsilon_1 \varepsilon_2 = \frac{1}{4k^2}\left(1 - \frac{3}{4k}\right)$

we get that

$$\text{Prob}\ \{1 \leq R\} < e^{-\alpha n} + n^{1 - (\frac{c}{4} - k)}$$

Let $d'(k) = (1-(c/4-k))(-1)$. Then $d'(k) > 1$ for

$$c(k) > 2 + \max\left(4k, \frac{\alpha + \log_e 4}{\varepsilon_1 \varepsilon_2}\right)$$

and

$$\text{Prob}\{1 \leqslant R\} \leqslant e^{-\alpha n} + n^{-d'(k)}$$

$$< 2n^{-d'(k)} \qquad \text{for large } n.$$

Hence

$$\text{Prob}\{R = 0\} > 1 - 2n^{-d'(k)} \quad . \qquad \text{QED}$$

7. k-STRONG BLOCKS FOR INTERMEDIATE EDGE DENSITIES

Let $c/n \leqslant p \leqslant c'(\log n/n)$. We wish to study the k-strong connectivity of this class of random digraphs.

Theorem 9. For any constant $k \geqslant 0$ and any $m = o(n)$ there is a constant $c_1(k) > 0$ and a function $t(n) > c_1(k)\log n/m$ such that if $p \geqslant t(n)/n$ then if X is the cardinality of the biggest k-strong block of $D_{n,p}$

$$\text{Prob}\{X \leqslant n-m\} \leqslant \frac{n^k}{e^{t(n)(m/2)}} \to 0 \qquad \text{as} \qquad n \to \infty \; .$$

Proof. Assume that in the instance D of $D_{n,p}$ the cardinality X of the biggest k-strong block satisfies the inequality $X \leqslant n-m$. Then we can find two sets A,B (as in Proof of Theorem 6) such that $|A| = 1/2\ m$, $|B| = n-km$ and no edge from A to B (or from B to A). This event is above bounded by the probability $1-q$, where

$q = \text{Prob}\{$for every pair of disjoint sets A,B of vertices of the above sizes, there is at least one edge from A to $B\}$. We shall show that $q \to 1$ as $n \to \infty$. Let us enumerate all possible pairs of sets of vertices of the above sizes. Call them

$$(A_1, B_1), (A_2,B_2),\ldots,(A_g, B_g)$$

where

$$g = \binom{n}{\frac{1}{2}m}\binom{n-m}{n-km} = \binom{n}{\frac{1}{2}m}\binom{n-m}{(k-1)m} .$$

We have by Baye's formula that

$$q = \text{Prob}\{E(A_1,B_1) \neq \emptyset \wedge \ldots \wedge E(A_g, B_g) \neq \emptyset\}$$

where $E(A,B)$ = set of edges from A to B.

So

$$q = \text{Prob}\, E(A_1,B_1) \neq \emptyset \;\; \text{Prob}\left\{\frac{E(A_2,B_2)\neq\emptyset}{E(A_1,B_1)\neq\emptyset}\right\} \cdots \text{Prob}\left\{\frac{E(A_g,B_g)\neq\emptyset}{\bigcap_{1}^{g-1} E(A_i,B_i)\neq\emptyset}\right\}$$

We need the following enumeration lemma:

<u>Lemma 5.</u> For every two sets A_i, B_i having at least one edge e from A_i to B_i, there are at least

$$g_1 = \binom{n-2}{\frac{1}{2}m-1}\binom{n-2-(m-1)}{(k-1)m-1}$$

pairs of sets of sizes $1/2\ m$, $n-km$ which also contain this edge.

This lemma can be proved easily by taking out the two vertices of e and enumerating.

Corollary. There is a suitable enumeration of the sets in the q product such that for every term i not equal to 1 the next (at least) g_1 terms (conditioned on the existence of an edge from A_i to B_i) will be equal to 1.

Hence the value of q is

$$q \geqslant [\text{Prob}\{\text{at least an edge from } A_1 \text{ to } B_1\}]^{g/g_1} .$$

But

$$\frac{g}{g_1} \leqslant \left(\frac{n}{m}\right)^k \qquad \text{as} \quad n \to \infty .$$

(In fact

$$\frac{g}{g_1} \to \left(\frac{n}{m}\right)^k \qquad \text{as} \quad n \to \infty .)$$

Hence,

$$q \geqslant \left[1 - (1-p)^{\frac{1}{2} m \cdot (n-km)}\right]^{(n/m)^k}$$

or

$$q \geqslant \left[1 - [(1-p)^{1/p}]^{p \frac{1}{2} m(n-km)}\right]^{(n/m)^k}$$

or

$$q \geqslant \left[1 - e^{-p \frac{m}{2} (n-km)}\right]^{\left(\frac{n}{m}\right)^k}$$

or

$$q \geqslant 1 - \left(\frac{n}{m}\right)^k e^{-\frac{t(n)m}{2}}$$

or

$$q \geqslant 1 - e^{-\left[\frac{t(n)m}{2} - k \log n\right]} > 1 - n^{-2} \qquad \text{if} \quad c_1(k) > 2k + 4.$$

(Since $1/2\ t(n)m > 1/2\ c_1(k)\log n > (k+2)\log n$ only if $c_1(k) > 2k + 4$.)

So,

$$\text{Prob}\{X < n - m\} < e^{-\left[\frac{t(n)m}{2} - k \log n\right]} \to 0 \qquad \text{as} \quad n \to \infty$$

for the above values of $c_1(k)$. QED

<u>Corollary</u>. For each $k > 0$, the digraph $D_{n,p}$ with $p \geqslant c_1(k)/n$ has a k-strong block of cardinality $> n - \log n$, with probability $> 1 - n^{-[c_1(k)1/2-k]}$.

<u>Proof</u>. Just set $m = \log n$ and $t(n) \geqslant c_1(k)$ in the previous theorem.

APPENDIX

LEMMA 1. The minimum number of vertices separating vertex s from vertex t in the direction s to t, is the maximum number of vertex disjoint s to t paths.

<u>Proof</u>. (A variation of Dirac's Proof for a version of Menger's Theorem.)

It is clear that if k points separate s from t then there can be no more than k disjoint paths from s to t.

It remains to show that if it takes k points to separate s and t (in the direction $s \to t$) in the digraph D, then there are k disjoint st paths in D. This is clearly true for $k = 1$. Assume it is not true for some $k > 1$. Let h be the smallest such k and let F be a digraph with the minimum number of vertices for which the theorem fails for h. We remove edges from F until we obtain a digraph D' such that h vertices are required to separate s,t (in the direction st) in D', but for any edge x in D', only $h-1$ vertices are required to separate s,t in $D'-x$. Let us investigate properties of this D'.

By definition of D', for every x edge of D', there is a set $S(x)$ of $h-1$ vertices separating s,t (in the st direction) in $D'-x$. Now, $D'-S(x)$ contains at least one st path, since it takes h vertices to separate s,t in D'. Each such st path must contain the edge $x = (u,v)$ since it is not a path in $D'-x$. So, $u,v \notin S(x)$ and if $u \neq s$, $u \neq t$ then $S(x) \cup \{u\}$ separates s from t (in the st direction) in D'.

If there is a vertex w such that $(s,w),(w,t)$ are edges in D', then $D'-w$ requires $h-1$ vertices to separate s,t and so it has $h-1$ disjoint st paths. Replacing w, we get h disjoint st paths in D'. So, we showed

(I) No such w exists in D'.

Let W be any collection of h vertices separating s from t (in st direction) in D'. An sW path is a path starting at s and ending in some $w_i \in W$ and containing no other vertex of W. Call the collection of all sW paths and Wt paths P_s and P_t, respectively. Then each st path begins with a member of P_s and ends with a member of P_t, because every such path contains a vertex of W. Moreover, the paths in P_s and P_t have the vertices of W and no others in common, since it is clear that each w_i is in at least one path in each collection and, if some other vertex were in both an sW and an Wt path then there would be an st path containing no vertex of W. Finally, either $P_s - W = \{s\}$ or $P_t - W = \{t\}$ since, if not, then both P_s plus the edges $\{(w_1,t),(w_2,t),\ldots\}$ and P_t plus the edges $\{(s,w_1),(s,w_2),\ldots\}$ are digraphs with fewer vertices than D' in which s,t are nonadjacent and h-connected and therefore in each there are h disjoint st paths. Combining the sW and Wt portions of these paths, we can construct h disjoint st paths in D', and thus have a contradiction. So

(II) Any collection W of h vertices separating s from t (to the st direction) has the property : $\forall u \in W$:
(s,u) is an edge
or (u,t) is an edge.

Now we complete the proof.

Let $P = \{(s,u_1),(u_1,u_2),\ldots,(*,t)\}$ be a shortest st path in D' and let $u_1u_2 = x$. By (I), $u_2 \neq t$.

Form $S(x) = \{u_1,u_2,\ldots,u_{h-1}\}$ as above, separating s from t in $D' - x$. By (I), $(u_1,t) \notin D'$, so by (II)

$$\text{with } W = S(x) \cup \{u_1\} \text{ we get } (s,u_i) \in D', \forall i.$$

Thus, by (I), $(u_i,t) \notin D'$, $\forall i$. However, if we pick $W = S(x) \cup \{u_2\}$ instead, we have by (II) that $(s,u_2) \in D'$, contradicting our choice of P as a shortest st path.

QED

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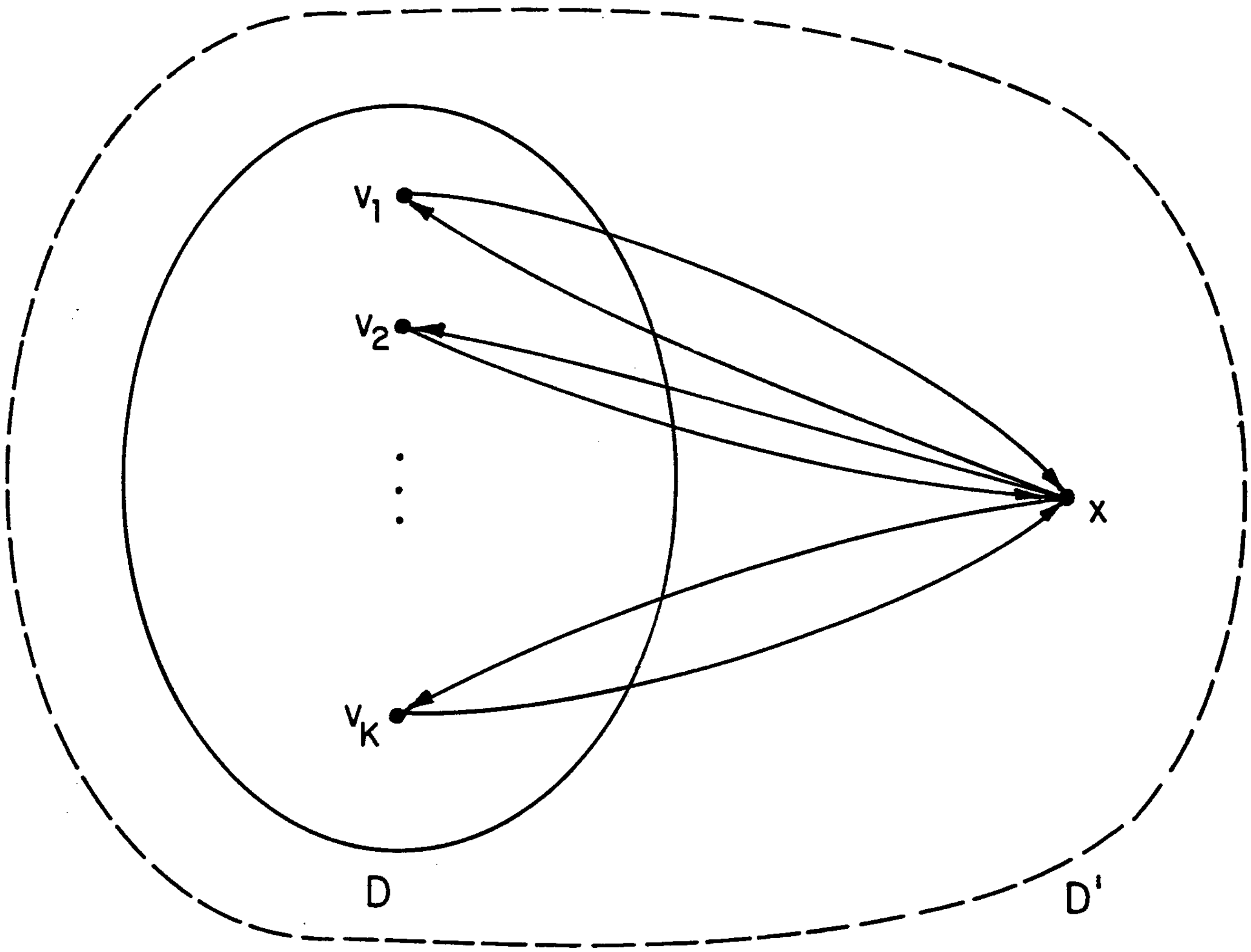

Figure 1

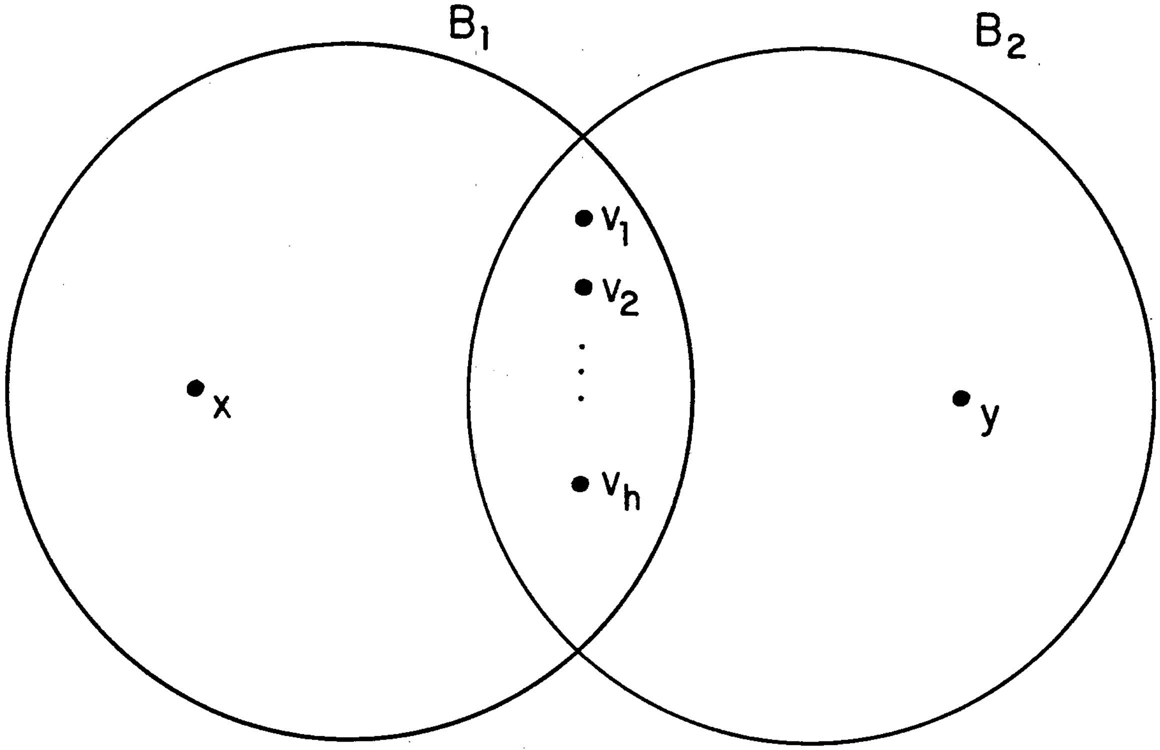

Figure 2