Definition: A language $L$ is **recursively enumerable** if there exists a TM $M$ such that $L = L(M)$.

if $w \in L$? if $w \notin L$?

Definition: A language $L$ is **recursive** if there exists a TM $M$ such that $L = L(M)$ and $M$ halts on every $w \in \Sigma^*$.

**Enumeration procedure for recursive languages**

To enumerate all $w \in \Sigma^*$ in a recursive language $L$:

- Let $M$ be a TM that recognizes $LTL = L(M)$.
- Construct 2-tape TM $M'$
  - Tape 1 will enumerate the strings in $\Sigma^*$
  - Tape 2 will enumerate the strings in $L$.
    - On tape 1 generate the next string $v$ in $\Sigma^*$
    - simulate $M$ on $v$
      - if $M$ accepts $v$ then write $v$ on tape 2.

**Enumeration procedure for recursively enumerable languages**

To enumerate all $w \in \Sigma^*$ in a recursively enumerable language $L$:

Repeat forever

- Generate next string (Suppose $k$ strings have been generated: $w_1, w_2, ..., w_k$)
- Run $M$ for one step on $w_k$
  - Run $M$ for two steps on $w_{k-1}$.
  - ...
  - Run $M$ for $k$ steps on $w_1$.
  - If any of the strings are accepted then write them to tape 2.

**Theorem** For any nonempty $\Sigma^*$ there exist languages that are not recursively enumerable.

**Proof:**
• A language is a subset of $\Sigma^*$.
  The set of all languages over $\Sigma$ is

**Theorem** There exists a recursively enumerable language $L$ such that $\overline{L}$ is not recursively enumerable.

**Proof:**

• Let $\Sigma = \{a\}$

Enumerate all TM’s over $\Sigma$:

|       | a | aa | aaa | aaaa | aaaaa | ...
|-------|---|----|-----|-------|-------|---|
| $L(M_1)$ | 0 | 1  | 1   | 0     | 1     | ...
| $L(M_2)$ | 1 | 0  | 1   | 0     | 1     | ...
| $L(M_3)$ | 0 | 0  | 1   | 1     | 0     | ...
| $L(M_4)$ | 1 | 1  | 0   | 1     | 1     | ...
| $L(M_5)$ | 0 | 0  | 0   | 1     | 0     | ...
| ...     |   |    |     |       |       |   |

The next two theorems in conjunction with the previous theorem will show that there are some languages that are recursively enumerable but not recursive.

**Theorem** If languages $L$ and $\overline{L}$ are both RE then $L$ is recursive.

**Proof:**

• There exists an $M_1$ such that $M_1$ can enumerate all elements in $L$.
  There exists an $M_2$ such that $M_2$ can enumerate all elements in $\overline{L}$.

To determine if a string $w$ is in $L$ or not in $L$ perform the following algorithm:
Theorem: If $L$ is recursive then $\bar{L}$ is recursive.

Proof:

- $L$ is recursive then there exists a TM $M$ such that $M$ can determine if $w$ is in $L$ or $w$ is not in $L$. $M$ outputs a 1 if a string $w$ is in $L$ and outputs a 0 if a string $w$ is not in $L$.

Construct TM $M'$ that does the following. $M'$ first simulates TM $M$. If TM $M$ halts with a 1 then $M'$ erases the 1 and writes a 0. If TM $M$ halts with a 0 then $M'$ erases the 0 and writes a 1.

Hierarchy of Languages:

Definition A grammar $G=(V,T,R,S)$ is *unrestricted* if all productions are of the form

$$u \rightarrow v$$

where $u \in (V \cup T)^+$ and $v \in (V \cup T)^*$

Example:

Let $G=(\{S,T,A,I,X\}|(a|b)|R|S)FR=$

$$S \rightarrow bAaX$$

$$bAa \rightarrow abA$$

$$AX \rightarrow \epsilon$$

Example Find an unrestricted grammar $G$ s.t. $L(G)=\{a^n b^n c^n | n > 0\}$

$G=(\{V,T,I,R\})$

$V=\{S,T,A,B,I,D,E,I,X\}$
\[ T = \{ \text{all} \text{ } \text{strings} \} \]

\[ R = \]

1) \( S \rightarrow AX \)
2) \( A \rightarrow aAbc \)
3) \( A \rightarrow aBbc \)
4) \( Bb \rightarrow bB \)
5) \( Bc \rightarrow D \)
6) \( Dc \rightarrow cD \)
7) \( Db \rightarrow bD \)
8) \( DX \rightarrow EXc \)
9) \( BX \rightarrow \epsilon \)
10) \( cE \rightarrow Ec \)
11) \( bE \rightarrow Eb \)
12) \( aE \rightarrow aB \)

There are some rules missing in the grammar.

To derive string \( \text{aaa}a\text{bb}b\text{ccc} \) use productions 1-2 and 3 to generate a string that has the correct number of a’s b’s and c’s. The a’s will all be together but the b’s and c’s will be intertwined.

\[ S \Rightarrow AX \Rightarrow aAbcX \Rightarrow aaAbcbcX \Rightarrow aaaaBbcbcbX \]

**Theorem** If \( G \) is an unrestricted grammar then \( L(G) \) is recursively enumerable.

**Proof:**

- List all strings that can be derived in one step.

List all strings that can be derived in two steps.

**Theorem** If \( L \) is recursively enumerable then there exists an unrestricted grammar \( G \) such that \( L = L(G) \).

**Proof:**

- \( L \) is recursively enumerable.
  \( \Rightarrow \) there exists a TM \( M \) such that \( L(M) = L \).

\[ M = (K, \Sigma, \Gamma, \delta, q_0, B, F) \]

\[ q_0 \delta w \vdash x_1 q_f x_2 \text{ for some } q_f \in F \Gamma x_1, x_2 \in \Gamma^* \]
Construct an unrestricted grammar $G$ s.t. $L(G)=L(M)$.

$S \Rightarrow w$

Three steps

1. $S \Rightarrow B \ldots B \#xqyB \ldots B$
   
   with $x\bar{y} \in \Gamma^*$ for every possible combination

2. $B \ldots B \#xqyB \ldots B \Rightarrow B \ldots B \#q_0wB \ldots B$

3. $B \ldots B \#q_0wB \ldots B \Rightarrow w$

**Definition** A grammar $G$ is context-sensitive if all productions are of the form

$$x \rightarrow y$$

where $x, y \in (V \cup T)^+$ and $|x| < |y|$

**Definition** $L$ is context-sensitive (CSL) if there exists a context-sensitive grammar $G$ such that $L=L(G)$ or $L=L(G) \cup \{\varepsilon\}$.

**Theorem** For every CSL $L$ not including $\varepsilon \Gamma \exists$ an LBA $M$ s.t. $L=L(M)$.

**Theorem** If $L$ is accepted by an LBA $M$ then $\exists$ CSG $G$ s.t. $L(M)=L(G)$.

**Theorem** Every context-sensitive language $L$ is recursive.

**Theorem** There exists a recursive language that is not CSL.