CPS 140 - Mathematical Foundations of CS
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Section: Context-Free Languages (handout)

Context-Free Languages (Read Ch. 3.1-3.2)

Regular languages:

- keywords in a programming language
- names of identifiers
- integers
- all misc symbols: ;

Not Regular languages:

- $\{a^n c b^n | n > 0\}$
- expressions - $((a + b) - c)$
- block structures - {} in C++

Definition: A grammar G=(V,Σ,R,S) is context-free if all productions are of the form

$$A \rightarrow x$$

Where $A \in V$ and $x \in (V \cup \Sigma)^*$.

Definition: L is a context-free language (CFL) iff ∃ context-free grammar (CFG) G s.t. L=L(G).

Example: G=({S},{a,b},R,S)

$$S \rightarrow aSb \mid ab$$

Derivation of aaabbb:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaabbb$$

L(G) =

Example: G=({S},{a,b},R,S)

$$S \rightarrow aSa \mid bSb \mid a \mid b \mid \epsilon$$

Derivation of ababa:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow ababa$$

$\Sigma = \{a, b\}$, L(G) =

Example: G=({S,A,B},{a,b,c},S,P)

$$\begin{aligned} &S \rightarrow AcB \\ &A \rightarrow aAa \mid \epsilon \\ &B \rightarrow Bbb \mid \epsilon \end{aligned}$$

L(G) =

Derivations of aacbb:

1. $S \Rightarrow \underline{A}cB \Rightarrow a\underline{A}acB \Rightarrow aac\underline{B} \Rightarrow aac\underline{B}bb \Rightarrow aacbb$
2. $S \Rightarrow Ac\underline{B} \Rightarrow Ac\underline{B}bb \Rightarrow \underline{A}cbb \Rightarrow a\underline{A}acbb \Rightarrow aacbb$
   Note: Next variable to be replaced is underlined.

Definition: Leftmost derivation - in each step of a derivation, replace the leftmost variable.

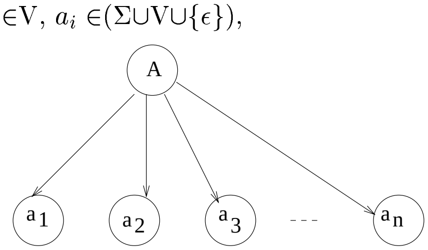
Definition: Rightmost derivation - in each step of a derivation, replace the rightmost variable.

Derivation Trees (also known as "parse trees")

A derivation tree represents a derivation but does not show the order productions were applied.

A derivation tree for G=(V,Σ, R, S):

- root is labeled S
- leaves labeled x, where x∈ $\Sigma \cup \{\epsilon\}$
- nonleaf vertices labeled A, A∈V
- For rule A→$a_1 a_2 a_3 \ldots a_n$, where A∈V, $a_i \in (\Sigma \cup V \cup \{\epsilon\})$,



Example: G=({S,A,B},{a,b,c},R,S)

$$\begin{array}{l} S \rightarrow AcB \\ A \rightarrow aAa \mid \epsilon \\ B \rightarrow Bbb \mid \epsilon \end{array}$$

Definitions Partial derivation tree - subtree of derivation tree.

If partial derivation tree has root S then it represents a sentential form.

Leaves from left to right in a derivation tree form the *yield* of the tree.

Yield (w) of derivation tree is such that w∈L(G).

The yield for the example above is

Example of partial derivation tree that has root S:

The yield of this example is ______________ which is a sentential form.

Example of partial derivation tree that does not have root S:

Membership Given CFG G and string w$\in \Sigma^*$, is w∈L(G)?

If we can find a derivation of w, then we would know that w is in L(G).

Motivation

G is grammar for C++.
w is C++ program.
Is w syntactically correct?

Example

G=({S}, {a,b}, R, S), R=

$$S \rightarrow SS \mid aSa \mid b \mid \epsilon$$

L_1=L(G) =

Is abbab ∈ L(G)?

Exhaustive Search Algorithm

For all i=1,2,3,...
  Examine all sentential forms yielded by i substitutions

Example: Is abbab $\in$ L(G)?

Theorem If CFG G does not contain rules of the form

$$A \to \epsilon$$
$$A \to B$$

where A,B$\in$V, then we can determine if w$\in$ L(G) or if w$\notin$ L(G).

- **Proof:** Consider
  1. length of sentential forms
  2. number of terminal symbols in a sentential form

Example: Let $L_2 = L_1 - \{\epsilon\}$. L_2=L(G) where G is:

$$S \to SS \mid aa \mid aSa \mid b$$

Show baaba $\notin$ L(G).

i=1
1. S $\Rightarrow$ SS
2. S $\Rightarrow$ aSa
3. S $\Rightarrow$ aa
4. S $\Rightarrow$ b

i=2
1. S $\Rightarrow$ SS $\Rightarrow$ SSS
2. S $\Rightarrow$ SS $\Rightarrow$ aSaS
3. S $\Rightarrow$ SS $\Rightarrow$ aaS
4. S $\Rightarrow$ SS $\Rightarrow$ bS
5. S $\Rightarrow$ aSa $\Rightarrow$ aSSa
6. S $\Rightarrow$ aSa $\Rightarrow$ aaSaa
7. S $\Rightarrow$ aSa $\Rightarrow$ aaaa
8. S $\Rightarrow$ aSa $\Rightarrow$ aba

Definition Simple grammar (or s-grammar) has all productions of the form:

$$A \to ax$$

where $A \in V$, $a \in \Sigma$, and $x \in V^*$ AND any pair (A,a) can occur in at most one rule.

Ambiguity

Definition: A CFG G is ambiguous if $\exists$ some $w \in L(G)$ which has two distinct derivation trees.

Example Expression grammar

$G=(\{E,I\}, \{a,b,+,*,(,)\}, R, E)$, R=

$$\begin{array}{l} E \to E+E \mid E*E \mid (E) \mid I \\ I \to a \mid b \end{array}$$

Derivation of a+b∗a is:

$E \Rightarrow \underline{E}+E \Rightarrow \underline{I}+E \Rightarrow a+\underline{E} \Rightarrow a+\underline{E}*E \Rightarrow a+\underline{I}*E \Rightarrow a+b*\underline{E} \Rightarrow a+b*\underline{I} \Rightarrow a+b*a$

Corresponding derivation tree is:

Another derivation of a+b∗a is:

$E \Rightarrow \underline{E}*E \Rightarrow \underline{E}+E*E \Rightarrow \underline{I}+E*E \Rightarrow a+\underline{E}*E \Rightarrow a+\underline{I}*E \Rightarrow a+b*\underline{E} \Rightarrow a+b*\underline{I} \Rightarrow a+b*a$

Corresponding derivation tree is:

Rewrite the grammar as an unambiguous grammar. (with meaning that multiplication has higher precedence than addition)

$$\begin{array}{l} E \to E+T \mid T \\ T \to T*F \mid F \\ F \to I \mid (E) \\ I \to a \mid b \end{array}$$

There is only one derivation tree for a+b∗c:

Definition If L is CFL and G is an unambiguous CFG s.t. L=L(G), then L is unambiguous.

Backus-Naur Form of a grammar:

- Nonterminals are enclosed in brackets <>
- For "$\rightarrow$" use instead "::="

Sample C++ Program:

```
int main ()
{
   int a;     int b;   int sum;
   a = 40;    b = 6;   sum = a + b;
   cout << "sum is "<< sum << endl;
   return 0;
}
```

"Attempt" to write a CFG for C++ in BNF (Note: <program> is start symbol of grammar.)

<program>	::= int main () <block>
<block>	::= { <stmt-list> }
<stmt-list>	::= <stmt> | <stmt><stmt-list> | <decl> | <decl><stmt-list>
<decl>	::= int <id> ; | double <id> ;
<stmt>	::= <asgn-stmt> | <cout-stmt> | <return-stmt>
<asgn-stmt>	::= <id> = <expr> ;
<expr>	::= <expr> + <expr> | <expr> $*$ <expr> | (<expr>) | <id>
<cout-stmt>	::= cout <out-list> ;
<return-stmt>	::= return <integer> ;

etc., Must expand all nonterminals!

So a derivation of the program test would look like:

<program> $\Rightarrow$ int main () <block>
$\Rightarrow$ int main () { <stmt-list> }
$\Rightarrow$ int main () { <decl> <stmt-list> }
$\Rightarrow$ int main () { int <id>; <stmt-list> }
$\Rightarrow$ int main () { int a ; <stmt-list> }
$\stackrel{*}{\Rightarrow}$ *complete C++ program*

More on CFG for C++

We can write a CFG G s.t. L(G)={syntactically correct C++ programs}.

But note that {semantically correct C++ programs} $\subset$ L(G).

Can't recognize redeclared variables:

Can't recognize if formal parameters match actual parameters in number and types: