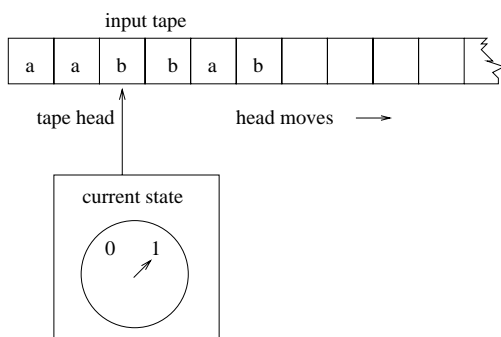
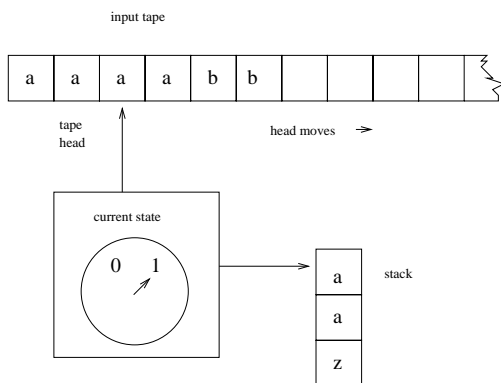


Pushdown Automata

A DFA = $(K, \Sigma, \delta, q_0, F)$



Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).



Definition: Nondeterministic PDA (NPDA) is defined by

$$M = (K, \Sigma, \Gamma, \Delta, q_0, z, F)$$

where

K is finite set of states

Σ is tape (input) alphabet

Γ is stack alphabet

q_0 is initial state

z is start stack symbol (bottom of stack marker)

$F \subseteq K$ is set of final states.

Δ is a finite subset of $(K \times (\Sigma \cup \{\epsilon\}) \times \Gamma^* \times K \times \Gamma^*)$

Example of transitions

$$\Delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \epsilon)\}$$

Meaning: If in state q_1 with “a” the current tape symbol and “b” the symbol on top of the stack, then pop “b”, and either

move to q_3 and push “b” on stack
move to q_4 and push “ab” on stack (“a” on top)
move to q_6

Transitions can be represented using a transition diagram.

The diagram for the above transitions is:

Each arc is labeled by a triple: $x,y;z$ where x is the current input symbol, y is the top of stack symbol which is popped from the stack, and z is a string that is pushed onto the stack.

Instantaneous Description (configuration):

$$(q, w, u)$$

Notation to describe the current state of the machine (q), unread portion of the input string (w), and the current contents of the stack (u).

Description of a Move:

$$(q_1, aw, bx) \vdash (q_2, w, yx) \\ \text{iff}$$

Definition Let $M = (K, \Sigma, \Delta, q_0, z, F)$ be a NPDA. $L(M) = \{w \in \Sigma^* \mid (q_0, w, z) \vdash^* (p, \epsilon, u), p \in F, u \in \Sigma^*\}$. The NPDA accepts all strings that start in q_0 and end in a final state.

Example: $L = \{a^n b^n \mid n \geq 0\}$, $\Sigma = \{a, b\}$, $\Delta = \{z, a\}$

Another Definition for Language Acceptance

NPDA M accepts $L(M)$ by empty stack:

$$L(M) = \{w \in \Sigma^* \mid (q_0, w, z) \vdash^* (p, \epsilon, \epsilon)\}$$

Example: $L = \{ww^R \mid w \in \Sigma^+\}$, $\Sigma = \{a, b\}$, $\Sigma = \{z, a, b\}$

Example: $L = \{ww \mid w \in \Sigma^*\}$, $\Sigma = \{a, b\}$

Examples for you to try on your own: (solutions are at the end of the handout).

- $L = \{a^n b^m \mid m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\delta = \{z, a\}$
- $L = \{a^n b^{n+m} c^m \mid n, m > 0\}$, $\Sigma = \{a, b, c\}$,
- $L = \{a^n b^{2n} \mid n > 0\}$, $\Sigma = \{a, b\}$

Theorem Given NPDA M that accepts by final state, \exists NPDA M' that accepts by empty stack s.t. $L(M) = L(M')$.

- **Proof** (sketch)

$$M = (K, \Sigma, \delta, q_0, z, F)$$

$$\text{Construct } M' = (K', \Sigma, \delta', q_s, z', F')$$

Theorem Given NPDA M that accepts by empty stack, \exists NPDA M' that accepts by final state.

- **Proof:** (sketch)

$$M = (K, \Sigma, \delta, q_0, z, F)$$

$$\text{Construct } M' = (K', \Sigma, \delta', q_s, z', F')$$

Theorem For any CFL L not containing ϵ , \exists an NPDA M s.t. $L=L(M)$.

- **Proof** (sketch)

Given (ϵ -free) CFL L .

$\Rightarrow \exists$ CFG G such that $L=L(G)$.

$\Rightarrow \exists G'$ in GNF, s.t. $L(G)=L(G')$.

$G'=(V,T,R,S)$. All productions in R are of the form:

Example: Let $G'=(V,T,R,S)$, $R=$

$S \rightarrow aSA \mid aAA \mid b$
 $A \rightarrow bBB$
 $B \rightarrow b$

Theorem Given a NPDA M , \exists a NPDA M' s.t. all transitions have the form $\Delta(q_i, a, A) = \{c_1, c_2, \dots, c_n\}$ where

$$\begin{array}{l} c_i = (q_j, \epsilon) \\ \text{or } c_i = (q_j, BC) \end{array}$$

Each move either increases or decreases stack contents by a single symbol.

- **Proof** (sketch)

Theorem If $L=L(M)$ for some NPDA M , then L is a CFL.

• **Proof:** Given NPDA M .

First, construct an equivalent NPDA M' that will be easier to work with. Construct M' such that

1. accepts if stack is empty
2. each move increases or decreases stack content by a single symbol. (can only push 2 variables or no variables with each transition)

$M'=(K,\Sigma,,\Delta,q_0,z,F)$

Construct $G=(V,\Sigma,R,S)$ where

$V=\{(q_i c q_j) | q_i, q_j \in K, c \in \Sigma, \}$

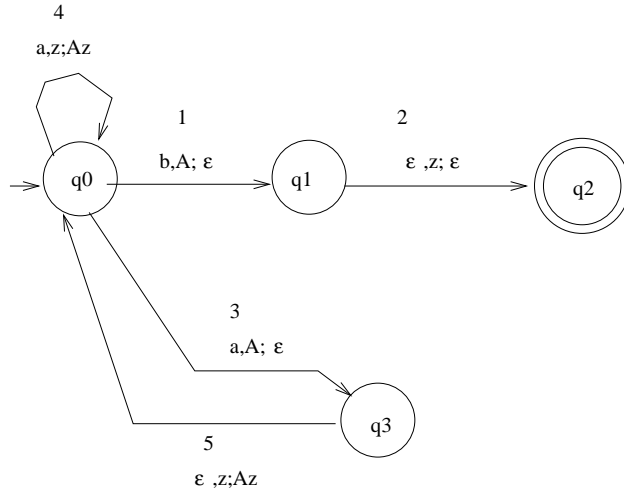
$(q_i c q_j)$ represents “starting at state q_i the stack contents are cw , $w \in \Sigma^*$, some path is followed to state q_j and the contents of the stack are now w ”.

Goal: $(q_0 z q_f)$ which will be the start symbol in the grammar.

Meaning: We start in state q_0 with z on the stack and process the input tape. Eventually we will reach the final state q_f and the stack will be empty. (Along the way we may push symbols on the stack, but these symbols will be popped from the stack).

Example:

$L(M) = \{aa^*b\}$, $M = (K, \Sigma, \Delta, q_0, z, F)$, $K = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, $\Delta = \{A, z\}$, $F = \{q_2\}$. M accepts by empty stack.



Construct the grammar $G = (V, T, R, S)$,

$$V = \{(q_0 A q_0), (q_0 z q_0), (q_0 A q_1), (q_0 z q_1), \dots\}$$

$$T = \Sigma$$

$$S = (q_0 z q_2)$$

R=

From transition 1 $(q_0 A q_1) \rightarrow b$

From transition 2 $(q_1 z q_2) \rightarrow \epsilon$

From transition 3 $(q_0 A q_3) \rightarrow a$

From transition 4

$(q_0 z q_0) \rightarrow$	$a(q_0 A q_0)(q_0 z q_0) $
	$a(q_0 A q_1)(q_1 z q_0) $
	$a(q_0 A q_2)(q_2 z q_0) $
	$a(q_0 A q_3)(q_3 z q_0) $
$(q_0 z q_1) \rightarrow$	$a(q_0 A q_0)(q_0 z q_1) $
	$a(q_0 A q_1)(q_1 z q_1) $
	$a(q_0 A q_2)(q_2 z q_1) $
	$a(q_0 A q_3)(q_3 z q_1) $
$(q_0 z q_2) \rightarrow$	$a(q_0 A q_0)(q_0 z q_2) $
	$a(q_0 A q_1)(q_1 z q_2) $
	$a(q_0 A q_2)(q_2 z q_2) $
	$a(q_0 A q_3)(q_3 z q_2) $
$(q_0 z q_3) \rightarrow$	$a(q_0 A q_0)(q_0 z q_3) $
	$a(q_0 A q_1)(q_1 z q_3) $
	$a(q_0 A q_2)(q_2 z q_3) $
	$a(q_0 A q_3)(q_3 z q_3) $

From transition 5

$(q_3 z q_0) \rightarrow$	$(q_0 A q_0)(q_0 z q_0) $
	$(q_0 A q_1)(q_1 z q_0) $
	$(q_0 A q_2)(q_2 z q_0) $
	$(q_0 A q_3)(q_3 z q_0) $
$(q_3 z q_1) \rightarrow$	$(q_0 A q_0)(q_0 z q_1) $
	$(q_0 A q_1)(q_1 z q_1) $
	$(q_0 A q_2)(q_2 z q_1) $
	$(q_0 A q_3)(q_3 z q_1) $
$(q_3 z q_2) \rightarrow$	$(q_0 A q_0)(q_0 z q_2) $
	$(q_0 A q_1)(q_1 z q_2) $
	$(q_0 A q_2)(q_2 z q_2) $
	$(q_0 A q_3)(q_3 z q_2) $
$(q_3 z q_3) \rightarrow$	$(q_0 A q_0)(q_0 z q_3) $
	$(q_0 A q_1)(q_1 z q_3) $
	$(q_0 A q_2)(q_2 z q_3) $
	$(q_0 A q_3)(q_3 z q_3) $

Recognizing aaab in M:

$(q_0, aaab, z)$	$\vdash (q_0, aab, Az)$
	$\vdash (q_3, ab, z)$
	$\vdash (q_0, ab, Az)$
	$\vdash (q_3, b, z)$
	$\vdash (q_0, b, Az)$
	$\vdash (q_1, \epsilon, z)$
	$\vdash (q_2, \epsilon, \epsilon)$

Derivation of string aaab in G:

$(q_0 z q_2)$	$\Rightarrow a(q_0 A q_3)(q_3 z q_2)$
	$\Rightarrow aa(q_3 z q_2)$
	$\Rightarrow aa(q_0 A q_3)(q_3 z q_2)$
	$\Rightarrow aaa(q_3 z q_2)$
	$\Rightarrow aaa(q_0 A q_1)(q_1 z q_2)$
	$\Rightarrow aaab(q_1 z q_2)$
	$\Rightarrow aaab$

Definition: A PDA $M=(K,\Sigma,,\delta,q_0,z,F)$ is *deterministic* if for every $q \in K$, $a \in \Sigma \cup \{\epsilon\}$, $b \in \Sigma$,

1. $\delta(q, a, b)$ contains at most 1 element
2. if $\delta(q, \epsilon, b) \neq \emptyset$ then $\delta(q, c, b)=\emptyset$ for all $c \in \Sigma$

Definition: L is DCFL iff \exists DPDA M s.t. $L=L(M)$.

Examples:

1. Previous pda for $\{a^n b^n | n \geq 0\}$ is deterministic.
2. Previous pda for $\{a^n b^m c^{n+m} | n, m > 0\}$ is deterministic.
3. Previous pda for $\{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\}$ is nondeterministic.

Note: There are CFL's that are not deterministic.

$L=\{a^n b^n | n \geq 1\} \cup \{a^n b^{2n} | n \geq 1\}$ is a CFL and not a DCFL.

- **Proof:** $L = \{a^n b^n : n \geq 1\} \cup \{a^n b^{2n} : n \geq 1\}$

It is easy to construct a NPDA for $\{a^n b^n : n \geq 1\}$ and a NPDA for $\{a^n b^{2n} : n \geq 1\}$. These two can be joined together by a new start state and ϵ -transitions to create a NPDA for L. Thus, L is CFL.

Now show L is not a DCFL. Assume that there is a deterministic PDA M such that $L = L(M)$. We will construct a PDA that recognizes a language that is not a CFL and derive a contradiction.

Construct a PDA M' as follows:

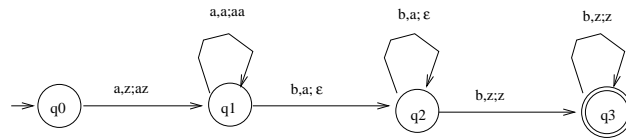
1. Create two copies of M : M_1 and M_2 . The same state in M_1 and M_2 are called cousins.
2. Remove accept status from accept states in M_1 , remove initial status from initial state in M_2 . In our new PDA, we will start in M_1 and accept in M_2 .
3. Outgoing arcs from old accept states in M_1 , change to end up in the cousin of its destination in M_2 . This joins M_1 and M_2 into one PDA. There must be an outgoing arc since you must recognize both $a^n b^n$ and $a^n b^{2n}$. After reading n b 's, must accept if no more b 's and continue if there are more b 's.
4. Modify all transitions that read a b and have their destinations in M_2 to read a c .

This is the construction of our new PDA.

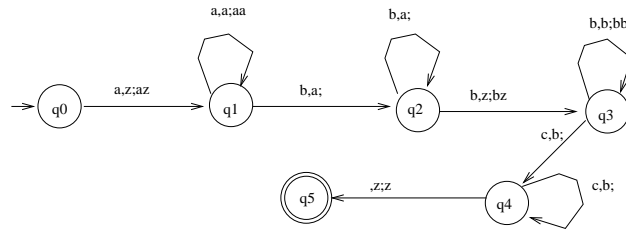
When we read $a^n b^n$ and end up in an old accept state in M_1 , then we will transfer to M_2 and read the rest of $a^n b^{2n}$. Only the b 's in M_2 have been replaced by c 's, so the new machine accepts $a^n b^n c^n$.

The language accepted by our new PDA is $a^n b^n c^n$. But this is not a CFL. Contradiction! Thus there is no deterministic PDA M such that $L(M) = L$. Q.E.D.

Example: $L = \{a^n b^m \mid m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\delta = \{z, a\}$



Example: $L = \{a^n b^{n+m} c^m \mid n, m > 0\}$, $\Sigma = \{a, b, c\}$,



Example: $L = \{a^n b^{2n} \mid n > 0\}$, $\Sigma = \{a, b\}$

