Homomorphism

Def. Let \( \Sigma_1, \Sigma_2 \) be alphabets. A homomorphism is a function

\[ h : \Sigma \rightarrow \Sigma^* \]

Example:

\[ \Sigma = \{a, b, c\}, \quad = \{0, 1\} \]

\[ h(a) = 11 \]
\[ h(b) = 00 \]
\[ h(c) = 0 \]

\[ h(bc) = \]
\[ h(ab^*) = \]

**Theorem** Let \( h \) be a homomorphism. If \( L \) is regular, then \( h(L) \) is regular.

**Example** using the homomorphism above.

\( L = a^*bb, h(L) = \)

Questions about regular languages:

\( L \) is a regular language.

- Given \( L, \Sigma, w \in \Sigma^* \), is \( w \in L \)?

- Is \( L \) empty?

- Is \( L \) infinite?

- Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = a^* b^*$
- $L_2 = \{a^n b^n | n > 0\}$

**Prove that** $L_2 = \{a^n b^n | n > 0\}$

- Proof: Suppose $L_2$ is regular.
**Pumping Lemma:** Let \( L \) be an infinite regular language. \( \exists \) a constant \( m > 0 \) such that any \( w \in L \) with \( |w| \geq m \) can be decomposed into three parts as \( w = xyz \) with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
x y^i z &\in L \quad \text{for all } i \geq 0
\end{align*}
\]

**Meaning:** Every long string in \( L \) (the constant \( m \) above corresponds to the finite number of states in \( M \) in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in \( L \).

**To Use the Pumping Lemma to prove \( L \) is not regular:**

- **Proof by Contradiction.**
  - Assume \( L \) is regular.
  - \( \Rightarrow \) \( L \) satisfies the pumping lemma.
  - Choose a long string \( w \) in \( L \), \( |w| \geq m \). (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  - Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m, |y| \geq 1 \) and \( x y^i z \in L \ \forall \ i \geq 0 \).
  - The pumping lemma does not hold. Contradiction!
  - \( \Rightarrow \) \( L \) is not regular. QED.

**Example** \( L = \{a^n b^n | n > 0 \} \)

\( L \) is not regular.

- **Proof:**
  - Assume \( L \) is regular.
  - \( \Rightarrow \) the pumping lemma holds.
  - Choose \( w = \) where \( m \) is the constant in the pumping lemma. (Note that \( w \) must be chosen such that \( |w| \geq m \).
  - The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with \( cb^m \). This is because of the restrictions \( |xy| \leq m \) and \( |y| > 0 \). So the partition is:

It should be true that \( x y^i z \in L \) for all \( i \geq 0 \).
**Example**  \( L = \{a^n b^{n+s} c^s \mid n, s > 0\} \)

L is not regular.

**Proof:**
Assume \( L \) is regular.
⇒ the pumping lemma holds.
Choose \( w = \)

The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with the rest of the string \( b^{m+s} c^s \).
This is because of the restrictions \( |xy| \leq m \) and \( |y| > 0 \). So the partition is:

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**Example**  \( \Sigma = \{a, b\}, L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\} \)

L is not regular.

**Proof:**
Assume \( L \) is regular.
⇒ the pumping lemma holds.
Choose \( w = \)

So the partition is:
Example \( L = \{a^3b^n c^{n-3} | n > 3\} \)

L is not regular.

- **Proof:**
  Assume \( L \) is regular. \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = a^3 b^m c^{m-3} \) where \( m \) is the constant in the pumping lemma. There are three ways to partition \( w \) into three parts, \( w = xyz. 1) y \) contains only \( a \)'s \( 2) y \) contains only \( b \)'s and \( 3) y \) contains \( a \)'s and \( b \)'s
  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide \( w \) into three parts s.t. the pumping lemma contraints were true).
  **Case 1:** \( y \) contains only \( a \)'s. Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and \( z \) contains 0 to 2 \( a \)'s concatenated with the rest of the string \( b^m c^{m-3} \), such that there are exactly 3 \( a \)'s. So the partition is:

  \[
  x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}
  \]

  where \( k \geq 0, j > 0, \) and \( k + j \leq 3 \) for some constants \( k \) and \( j \).
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  \( xy^2 z = (x)(y)(y)(z) = (a^k)(a^j)(a^{3-k-j} b^m c^{m-3}) = a^{3+k} b^m c^{m-3} \notin L \) since \( j > 0 \), there are too many \( a \)'s. Contradiction!

  **Case 2:** \( y \) contains only \( b \)'s Then \( x \) contains 3 \( a \)'s followed by 0 or more \( b \)'s, \( y \) contains 1 to \( m-3 \) \( b \)'s, and \( z \) contains 3 to \( m-3 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

  \[
  x = a^3 b^k \quad y = b^j \quad z = b^{m-k} c^{m-3}
  \]

  where \( k \geq 0, j > 0, \) and \( k + j \leq m-3 \) for some constants \( k \) and \( j \).
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  \( xy^2 z = a^3 b^{m-j} c^{m-3} \notin L \) since \( j > 0 \), there are too few \( b \)'s. Contradiction!

  **Case 3:** \( y \) contains \( a \)'s and \( b \)'s Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and 1 to \( m-3 \) \( b \)'s, \( z \) contains 3 to \( m-1 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

  \[
  x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j} c^{m-3}
  \]

  where \( 3 \geq k > 0, \) and \( m-3 \geq j > 0 \) for some constants \( k \) and \( j \).
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  \( xy^2 z = a^3 b^i a^k b^m c^{m-3} \notin L \) since \( j, k > 0 \), there are \( b \)'s before \( a \)'s. Contradiction!
  \( \Rightarrow \) There is no partition of \( w \).
  \( \Rightarrow \) \( L \) is not regular! QED.
**To Use Closure Properties** to prove \( L \) is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**
  
  Assume \( L \) is regular.
  
  Apply closure properties to \( L \) and other regular languages, constructing \( L' \) that you know is not regular.
  
  closure properties \( \Rightarrow L' \) is regular.
  
  Contradiction!
  
  \( L \) is not regular. QED.

**Example** \( L = \{a^3b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  
  Assume \( L \) is regular.
  
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  
  \( h(a) = a \quad h(b) = a \quad h(c) = b \)
  
  \( h(L) = \)
Example \( L = \{ a^m b^n a^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.

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**Example:** \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.

- **Proof:**
  Assume \( L_1 \) is regular.
  Goal is to try to construct \( \{ a^n b^n | n > 0 \} \) which we know is not regular.
  Let \( L_2 = \{ a^n \} \). \( L_2 \) is regular.
  By closure under right quotient, \( L_3 = L_1 \setminus L_2 = \{ a^n b^n a^n | 0 \leq p \leq n, n > 0 \} \) is regular.
  By closure under intersection, \( L_4 = L_3 \cap \{ a^* b^* \} = \{ a^n b^n | n > 0 \} \) is regular.
  Contradiction, already proved \( L_4 \) is not regular!
  Thus, \( L_1 \) is not regular. QED.