1. (8 pts) Consider the following two sets and then answer the questions. Note that $2^B$ is notation for the power set of $B$.

\[ A = \{1, 4\} \quad B = \{4, 6\} \]

(a) $A \cap B = \{4\}$
(b) $A \times B = \{(1, 4), (1, 6), (4, 4), (4, 6)\}$
(c) $2^B = \{\emptyset, \{4\}, \{6\}, \{4, 6\}\}$
(d) $2^B \cap A = \emptyset$

2. (14 pts) Answer TRUE or FALSE to each of the questions below.

(a) Given a regular grammar $G$, there exists an NFA $M$ with just one final state such that $L(G) = L(M)$. (TRUE or FALSE?) TRUE
(b) Given a regular expression $r$, there exists a CFG $G$ such that $L(r) = L(G)$. (TRUE or FALSE?) TRUE
(c) $L = \{w \in \Sigma^* \mid n_a(w) \text{ mod } 3 = 0, \text{ and } n_b(w) \text{ mod } 5 = 0\}$. $L$ is regular. (TRUE or FALSE?) TRUE
(d) $L = \{a^n b^m \mid n + m > 5, n > 0, m \geq 0\}$. $L$ is regular. (TRUE or FALSE?) TRUE
(e) $L = \{a^n b^m c^p \mid n > 0, m \geq 0, p > 0\}$. $L$ is regular. (TRUE or FALSE?) TRUE
(f) $L = \{a^n b^m \mid m > 5n, n > 0\}$. $L$ is regular. (TRUE or FALSE?) FALSE
(g) $L = \{a^n \mid n/10 \text{ is an integer}\}$. $L$ is regular. (TRUE or FALSE?) TRUE

3. (4 pts) If $L_1$ is not regular and $L_1 \cap L_2$ is not regular, then $L_2$ is not regular. (TRUE or FALSE?). Explain.

FALSE. Consider $L_1 = \{a^n b^n \mid n > 0\}$ and $L_2 = a^* b^*$, then $L_1 \cap L_2 = L_1$, $L_1$ is not regular and $L_2$ is regular.
4. (14 pts) Draw DFA’s for the following languages. Do not show trap states.

(a) \( L = \{ w \in \Sigma^* | w \text{ has at least one } a \text{ and at least one } b \}, \Sigma = \{ a, b \} \). 

(b) \( L = \{ w \in \Sigma^* | \text{the substring } bab \text{ does not appear in } w \}, \Sigma = \{ a, b \} \).
5. (8 pts) Convert the following NFA to a DFA using the algorithm discussed in class.
6. (8 pts) Convert the following DFA to a minimum state DFA.
7. (6 pts) Give a regular expression for the following NFA.

\[ b^*aa(ba+bb)^* \]

8. (14 pts) Give a context-free grammar for each of the following languages.

(a) \( L = \{ a^n b^m c^p \mid m = n + p, n > 0, p \geq 0 \} \).

\[
S \rightarrow AB \\
A \rightarrow aAb \mid ab \\
B \rightarrow bBc \mid \lambda
\]

(b) \( L = \{ a^m b^{2m} c^{2n} \mid m > 0, n \geq 0 \} \).

\[
S \rightarrow AB \\
A \rightarrow aAb \mid ab \\
B \rightarrow cBc \mid \lambda
\]

9. (8 pts) Here is the Theorem (and algorithm) for removing left recursion from a grammar.

- **Theorem** (Removing Left recursion) Let \( G = (V, T, S, P) \) be a CFG. Divide productions for variable \( A \) into left-recursive and non left-recursive productions:

\[
A \rightarrow Ax_1 \mid Ax_2 \mid \ldots \mid Ax_n \\
A \rightarrow y_1 y_2 \ldots y_m
\]

where \( x_i, y_i \) are in \( (V \cup T)^* \).

Then \( G' = (V \cup \{Z\}, T, S, P') \) and \( P' \) replaces rules of form above by

\[
A \rightarrow y_i y_i Z, \text{ i=1,2,\ldots,m} \\
Z \rightarrow x_i x_i Z, \text{ i=1,2,\ldots,n}
\]

Using the method above, transform the following grammar into a grammar with no left recursion. DO NOT simplify the grammar.

\[
S \rightarrow Ac \mid Sab \mid b \\
A \rightarrow Baa \mid a \\
B \rightarrow Bb \mid Bac \mid b
\]
Grammar with no left recursion is:

\[
\begin{align*}
S & \rightarrow Ac \mid b \mid AcZ \mid bZ \\
Z & \rightarrow ab \mid abZ \\
A & \rightarrow Baa \mid a \\
B & \rightarrow bX \mid b \\
X & \rightarrow b \mid ac \mid bX \mid acX
\end{align*}
\]

10. (8 pts) Use the Pumping Lemma (shown below) to prove

\[L = \{w \in \Sigma^* \mid n_a(w) = 2 \cdot n_b(w)\}\] is not regular.

**Pumping Lemma:** Let \(L\) be an infinite regular language. \(\exists\) a constant \(m > 0\) such that any \(w \in L\) with \(|w| \geq m\) can be decomposed into three parts as \(w = xyz\) with

\[
\begin{align*}
|xy| & \leq m \\
|y| & \geq 1 \\
x y^i z & \in L \quad \text{for all } i \geq 0
\end{align*}
\]

- Proof: (SHOW ALL STEPS! Some have been started for you.)

  Assume \(L\) is regular.

  Choose \(w = a^{2m} b^n\)

  Show there is no way to partition this string \(w=xyz\) such that the properties of the pumping lemma hold.

  A representative of all possible partitions for \(y = a^k, \ k > 0\)

  Thus, \(x = a^j\), and \(z = a^{2m-k-j} b^n\)

  Let \(i=2\), \(x y^i z = a^{2m+k} b^n \notin L\), since \(n_a > 2 \cdot n_b\).

  Contradiction! Thus, \(L\) is not regular.
11. (8 pts) Consider the following property:

\[ 2\text{shorter}(L) = \{z \mid xz \in L \text{ and } |x| = 2\}, \ x, z \in \Sigma^*, \Sigma = \{a, b\}. \]

Prove that the regular languages are closed under the \(2\text{shorter}\) property.

\(L\) is regular, so there exists an NFA for \(L\), called \(M\). Construct an NFA for \(2\text{shorter}(L)\) by adding a new start state and have lambda arcs to all states that are reachable by a path of length 2 from the original start state in \(M\). The resulting NFA recognizes \(2\text{shorter}(L)\), Thus \(2\text{shorter}(L)\) is a regular language.